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Some Results on Cordial Related Labelings of The Flower Snark Graph

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Abstract. This work is proposed to derive some results on the Flower Snark graph J_n in view of the fulfillment of the Total Edge Product Cordial, and some special cases of k - product Cordial labelings. A graph G is said to be a Total Edge Product Cordial graph if an edge labeling function $f: E(G) \rightarrow \{0, 1\}$ induces a vertex labeling function $f^*: V(G) \rightarrow \{0, 1\}$ defined as the product of the labels of all the edges incident upon that vertex; and satisfying the condition $\left|\{v_f(0) + e_f(0)\} - \{v_f(1) + e_f(1)\}\right| \leq 1$. Also, a graph G is called a k-product Cordial graph if a vertex labeling function $f : V(G) \rightarrow \{0, 1, 2, \dots, k-1\}$ induces an edge labeling function $f^*: E(G) \rightarrow \{0, 1, 2, \dots, k-1\}$ which defines the labeling of an edge $e = uv$ as the product of the labels of the vertices u and v; modulo k satisfying the conditions $\left|v_f(i) - v_f(j)\right| \leq 1$ and $\left|e_f(i) - e_f(j)\right| \leq 1$, where $0 \leq i, j \leq k-1$.
AMS Subject Classification. 05C78

Keywords. Edge Product Cordial labeling, k-Product Cordial labeling, Flower Snark Graph.

1. INTRODUCTION

Labeling of graphs is an innovative area of study within graph theory, which involves assigning labels, usually numbers, to the vertices or the edges or both of a graph subject to certain mathematical rules. The concept of Cordial labeling was introduced by I. Cahit in the year 1987. The idea of Cordial labeling is a refinement of the concept of graceful labeling, which was introduced by Alexander Rosa in the year 1967. The E-Cordial labeling was introduced by R. Yilmaz and I. Cahit in the year 1997. This labeling emerged as a combined version of Cordial labeling and Edge Graceful labeling which were introduced by I. Cahit and S. Lo respectively. The notion of the product Cordial labeling was introduced by Sundaram, Ponraj and Soma Sundaram, which is a binary labeling. The k-product Cordial labeling is a generalized extension of the product Cordial labeling with k choices for labels instead of the two binaries. The proposed work surrounds these labeling patterns when investigated pertaining to the Flower Snark graph. Of the four results produced, three of them prove the non-admission of special cases of k-Product Cordial labelings for the Flower Snark graph. Such results of negation are a rarity in the domain of graph labeling.

2. DEFINITIONS

2.1 Cordial Labeling

Consider a graph $G = (V, E)$. Consider a vertex labeling function $f: V(G) \rightarrow \{0, 1\}$ which induces an edge labeling function $f^*: E(G) \rightarrow \{0, 1\}$ defined as $f^*(uv) = |f(u) - f(v)|$. Such a labeling function f is called a Cordial labeling of graph G, if it satisfies the conditions $\left|v_f(0) - v_f(1)\right| \leq 1$ and $\left|e_f(0) - e_f(1)\right| \leq 1$; where $v_f(i)$ denotes the number of vertices of G labeled using f and $e_f(i)$ denotes the number of edges of G labeled using f^*; $i \in \{0, 1\}$. A graph which admits a Cordial labeling is called a Cordial graph.

2.2 Product Cordial Labeling

Consider a graph $G = (V, E)$. Consider a vertex labeling function $f: V(G) \rightarrow \{0, 1\}$ which induces an edge labeling function $f^*: E(G) \rightarrow \{0, 1\}$ defining the labeling of an edge $e = uv$ as the product of the labels of the vertices u and v. Such a labeling function f is called a Product Cordial labeling of graph G, if it satisfies the conditions $\left|v_f(0) - v_f(1)\right| \leq 1$ and $\left|e_f(0) - e_f(1)\right| \leq 1$; where $v_f(i)$ denotes the number of vertices of G labeled using f and $e_f(i)$ denotes the number of edges of G labeled using f^*; $i \in \{0, 1\}$. A graph which admits a Product Cordial labeling is called a Product Cordial graph.

2.3 Edge Product Cordial Labeling

Consider a graph $G = (V, E)$. Consider an edge labeling function $f: E(G) \rightarrow \{0, 1\}$ which induces a vertex labeling function $f^*: V(G) \rightarrow \{0, 1\}$ defining the labeling of a vertex v as the product of the labels of all the edges incident on v. Such a labeling function f is called a Total Edge Product Cordial labeling of graph G, if it satisfies the conditions $\left|v_f(0) - v_f(1)\right| \leq 1$ and $\left|e_f(0) - e_f(1)\right| \leq 1$; where $v_f(i)$ denotes the number of vertices of G labeled using f^* and $e_f(i)$ denotes the number of edges of G labeled using f; $i \in \{0, 1\}$. A graph which admits a Total Edge Product Cordial labeling is called a Total Edge Product Cordial graph.

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2.4 Total Edge Product Cordial Labeling

Consider a graph $G = (V, E)$. Consider an edge labeling function $f: E(G) \rightarrow \{0, 1\}$ which induces a vertex labeling function $f^*: V(G) \rightarrow \{0, 1\}$ defining the labeling of a vertex v as the product of the labels of all the edges incident on v. Such a labeling function f is called a Total Edge Product Cordial labeling of graph G, if it satisfies the condition $\left|\{v_f(0) + e_f(0)\} - \{v_f(1) + e_f(1)\}\right| \leq 1$; where $v_f(i)$ denotes the number of vertices of G labeled using f^* and $e_f(i)$ denotes the number of edges of G labeled using f; $i \in \{0, 1\}$. A graph which admits a Total Edge Product Cordial labeling is called a Total Edge Product Cordial graph.

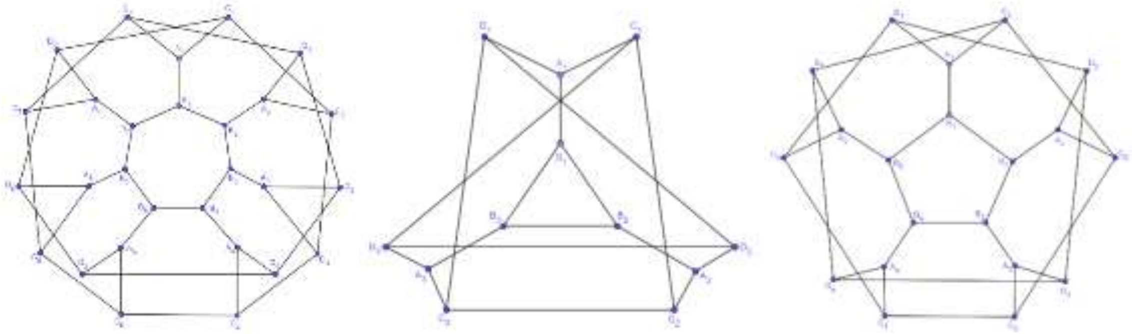
2.5 *k*-Product Cordial Labeling

Consider a graph $G = (V, E)$. Consider a vertex labeling function $f : V(G) \rightarrow \{0, 1, 2, \dots, k-1\}$ which induces an edge labeling function $f^*: E(G) \rightarrow \{0, 1, 2, \dots, k-1\}$ which defines the labeling of an edge $e = uv$ as the product of the labels of the vertices u and v; modulo k. Such a labeling function f is called a k-Product Cordial labeling of graph G, if it satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $\left|e_f(i) - e_f(j)\right| \leq 1$; where $v_f(i)$ denotes the number of vertices of G labeled using f and $e_f(i)$ denotes the number of edges of G labeled using f^*; $i, j \in \{0, 1, 2, \dots, k-1\}$. A graph which admits a k-Product Cordial labeling is called a k-Product Cordial graph.

2.6 Flower Snark Graph

A flower Snark J_n is a Snark graph with $4n$ vertices and $6n$ edges built using n copies of the star graph on 4 vertices where n is odd.

Construction : Denote the central vertices of each star as A_i, and the outer vertices are denoted as B_i, C_i and D_i. This results in a disconnected graph on $4n$ vertices with $3n$ edges A_iB_i, A_iC_i and A_iD_i for $1 \leq i \leq n$. Now construct the n-cycle $B_1B_2 \dots B_nB_1$. Then construct the $2n$-cycle $C_1C_2 \dots . C_nD_1D_2 \dots . D_nC_1$.



J_3 J_5 J_7

Figure 2.6.1

3. MAIN REULTS

Theorem 3.1 : A Flower Snark J_n is a Total Edge Product Cordial graph.

Proof :

Consider a Flower Snark J_n with $4n$ vertices and $6n$ edges built using n copies of the star graph on 4 vertices, where n is odd. Consider the Vertex set and the Edge set of J_n respectively as

$V = \{A_i, B_i, C_i\ D_i\ ; 1 \leq i \leq n\}$ and

$E = \{A_iB_i, A_iC_i, A_iD_i; 1 \leq i \leq n\} \cup \{B_iB_{i+1}, C_iC_{i+1}, D_iD_{i+1}\ ; 1 \leq i \leq n-1\} \cup \{B_1B_n, C_1D_n, D_1C_n\}$

Thus,

$|V| = n + n + n + n = 4n$ and

$|E| = (n + n + n) + (n - 1 + n - 1 + n - 1) + (1 + 1 + 1) = 3n + (3n - 3) + 3 = 6n$

Now the edge labeling function is defined as

$f(A_iB_i) = 1;\ 1 \leq i \leq n$

$f(B_1B_n) = 1$

$f(B_iB_{i+1}) = 1, 1 \leq i \leq n - 1$

$f(A_iC_i) = f(A_iD_i) = 1\ ; 1 \leq i \leq \dfrac{n+1}{2}$

From the remaining edges, assign label 1 to any $\left(\frac{n-3}{2}\right)$ non adjacent edges as label 1.

Thus, the total number of edges in J_n with label 1 is

$$e_f(1) = (n) + (n) + 2\left(\frac{n+1}{2}\right) + \left(\frac{n-3}{2}\right)$$
$$= \frac{7n-1}{2}$$

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So, the total number of edges in J_n with label 0 is

$$e_f(0) = 6n - \frac{7n-1}{2}$$
$$= \frac{5n+1}{2}$$

The above edge labeling function f will produce the following vertices with label 1 by the definition of the induced vertex labeling function f^*

$$f^*(A_i) = 1, 1 \le i \le \frac{n+1}{2}$$
$$f^*(B_i) = 1, 1 \le i \le n$$

Thus, the total number of vertices in J_n with label 1 is

$$v_f(1) = \frac{n+1}{2} + n$$
$$= \frac{3n+1}{2}$$

So, the total number of vertices in J_n with label 0 is

$$v_f(0) = 4n - \frac{3n+1}{2}$$
$$= \frac{5n-1}{2}$$

We now check the condition for Total Edge Product Cordiality as under.

$$\left|\{v_f(0) + e_f(0)\} - \{v_f(1) + e_f(1)\right|$$
$$= \left|\left\{\left(\frac{5n-1}{2}\right) + \left(\frac{5n+1}{2}\right)\right\} - \left\{\left(\frac{3n+1}{2}\right) + \left(\frac{7n-1}{2}\right)\right\}\right|$$
$$= \left|\left\{\left(\frac{5n-1}{2}\right) + \left(\frac{5n+1}{2}\right)\right\} - \left\{\left(\frac{3n+1}{2}\right) + \left(\frac{7n-1}{2}\right)\right\}\right|$$
$$= |5n - 5n\,|$$
$$= 0 \le 1$$

Thus, J_n admits the Total Edge Product Cordial labeling property.
Thus, J_n is a Total Edge Product Cordial graph.

Illustration 3.1.1 : Total Edge Product Cordial labeling of a Flower Snark J_7 is shown below.

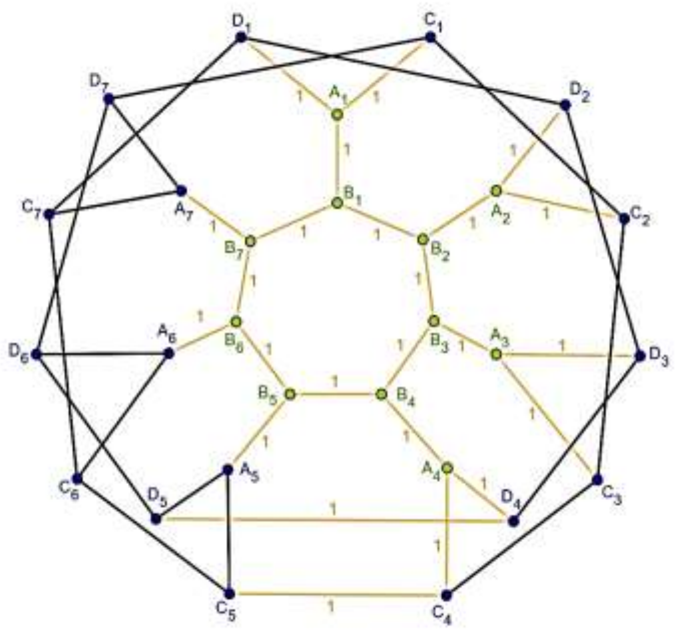

Fig 3.1.1

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$f(A_iB_i) = 1;\ 1 \le i \le 7$
$f(B_1B_n) = 1$
$f(B_iB_{i+1}) = 1, 1 \le i \le 6$

$f(A_iC_i) = f(A_iD_i) = 1\ ; 1 \le i \le 4$
From the remaining edges, we can assign label 1 to any $\left(\frac{n-3}{2}\right) = 2$ non adjacent edges as label 1.
Let those edges in this case be C_4D_4 and C_5D_5 as shown above.
Thus, the total number of edges in J_n with label 1 is
$e_f(1) = 7 + 1 + 6 + 8 + 2 = 24$
So, the total number of edges in J_7 with label 0 is
$e_f(0) = 42 - 24 = 18$
$$= \frac{5n+1}{2}$$
The above edge labeling function f will produce the following vertices with label 1 by the definition of the induced vertex labeling function f^*
$f^*(A_i) = 1, 1 \le i \le 4$
$f^*(B_i) = 1, 1 \le i \le 7$
Thus, the total number of vertices in J_n with label 1 is
$v_f(1) = 11$
So, the total number of vertices in J_7 with label 0 is
$v_f(0) = 28 - 11 = 17$
We now check the condition for Total Edge Product Cordiality as under.
$\left|\{v_f(0) + e_f(0)\} - \{v_f(1) + e_f(1)\right|$
$= |(17 + 18) - (11 + 24)|$
$= |35 - 35|$
$= 0 \le 1$
Thus, J_7 admits the Total Edge Product Cordial labeling property.
Thus, J_7 is a Total Edge Product Cordial graph.

Theorem 3.2 : A Flower Snark J_n is not a 3-Product Cordial graph.
Proof :
Consider a Flower Snark J_n with $4n$ vertices and $6n$ edges built using n copies of the star graph on 4 vertices, where n is odd. Consider the Vertex set and the Edge set of J_n respectively as
$V = \{A_i, B_i, C_i\ D_i\ ; 1 \le i \le n\}$ and
$E = \{A_iB_i, A_iC_i, A_iD_i\ ; 1 \le i \le n\} \cup \{B_iB_{i+1}, C_iC_{i+1}, D_iD_{i+1}\ ; 1 \le i \le n-1\} \cup \{B_1B_n, C_1D_n, D_1C_n\}$
Thus,
$|V| = n + n + n + n = 4n$ and
$|E| = (n + n + n) + (n - 1 + n - 1 + n - 1) + (1 + 1 + 1) = 3n + (3n - 3) + 3 = 6n$
In order to check for the 3-Product Cordiality, there are three choices from {0, 1, 2} for the labels of the vertices and the edges of the chosen graph. We choose the minimum number of vertices to assign label 0, in order to minimize the number of edges getting label 0, whose probability is 3 in 4 possible outcomes as product of labels 0 and 1. Moreover, these vertices are chosen in maximum adjacency for the same reason.
Thus, $v_0 = \left\lfloor\frac{4n}{3}\right\rfloor$ --- (1)
Also, the required number of edges getting label 0 is
$e_{0_R} = \frac{6n}{3} = 2n$ --- (2)
In order to break down the integer function expression, we consider the following three cases for the values v_0 depending on the order n of J_n, which is always odd.

Case 1 : $n = 3k$; $k = 1, 3, 5, 7, \ldots$
Thus, from (1), we have
$$v_0 = \left\lfloor\frac{4(3k)}{3}\right\rfloor = 4k = \frac{4n}{3} = n + \frac{n}{3}$$
In order to assign label 0 to these many vertices in the maximum possible adjacency, we choose the n vertices of the cycle $C_n : B_1B_2 \ldots B_nB_1$, and the remaining $\frac{n}{3}$ vertices are chosen to be the consecutive apices $A_i\ ;\ 1 \le i \le \frac{n}{3}$.

Thus, the vertex labeling function f is defined as

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$f(B_i) = 0, 1 \leq i \leq n$ and
$f(A_i) = 0, 1 \leq i \leq \frac{n}{3}$
This vertex labeling function f,will produce the following edges with label 0 as per the induce edge labeling function f^*.
$f^*(B_iB_{i+1}) = 0\ ; 1 \leq i \leq n-1$
$f^*(B_1B_n) = 0$
$f^*(A_iB_i) = 0\ ;\ 1 \leq i \leq n$
$f^*(A_iC_i) = f^*(A_iD_i) = 0\ ;\ 1 \leq i \leq \frac{n}{3}$
Thus, the resultant number of edges getting label 0 is
$e_0 = (n-1) + (1) + (n) + 2\left(\frac{n}{3}\right)$
$= \frac{8n}{3}$
which is always greater than $e_{0_R} = 2n$ for all $k = 1, 3, 5, 7, \ldots$
$\therefore e_0 > e_{0_R}$ for all k.
Thus, J_n does not admit a 3-Cordial labeling in this case.

Case 2 : $n = 3k + 2$; $k = 1, 3, 5, 7, \ldots$
Thus, from (1), we have
$v_0 = \left\lfloor \frac{4(3k+2)}{3} \right\rfloor = \left\lfloor \frac{12k+8}{3} \right\rfloor = \left\lfloor 4k + \frac{8}{3} \right\rfloor = 4k + 2 = (3k+2) + (k) = (n) + \left(\frac{n-2}{3}\right)$
In order to assign label 0 to these many vertices in the maximum possible adjacency, we choose the n vertices of the cycle $C_n : B_1B_2 \ldots B_nB_1$, and the remaining $\frac{n-2}{3}$ vertices are chosen to be the consecutive apices $A_i\ ;\ 1 \leq i \leq \frac{n-2}{3}$.
Thus, the vertex labeling function f is defined as
$f(B_i) = 0, 1 \leq i \leq n$ and
$f(A_i) = 0, 1 \leq i \leq \frac{n-2}{3}$
This vertex labeling function f,will produce the following edges with label 0 as per the induce edge labeling function f^*.
$f^*(B_iB_{i+1}) = 0\ ; 1 \leq i \leq n-1$
$f^*(B_1B_n) = 0$
$f^*(A_iB_i) = 0\ ;\ 1 \leq i \leq n$
$f^*(A_iC_i) = f^*(A_iD_i) = 0\ ;\ 1 \leq i \leq \frac{n-2}{3}$
Thus, the resultant number of edges getting label 0 is
$e_0 = (n-1) + (1) + (n) + 2(\frac{n-2}{3})$
$= \frac{8n-4}{3}$
which is always greater than $e_{0_R} = 2n$ for all $k = 1, 3, 5, 7, \ldots$
$\therefore e_0 > e_{0_R}$ for all k.
Thus, J_n does not admit a 3-Cordial labeling in this case.

Case 3 : $n = 3k - 2$; $k = 3, 5, 7, \ldots$
Thus, from (1), we have
$v_0 = \left\lfloor \frac{4(3k-2)}{3} \right\rfloor = \left\lfloor \frac{12k-8}{3} \right\rfloor = \left\lfloor 4k - \frac{8}{3} \right\rfloor = (4k-2) - 1 = (3k-2) + (k-1) = (n) + \left(\frac{n-1}{3}\right)$
In order to assign label 0 to these many vertices in the maximum possible adjacency, we choose the n vertices of the cycle $C_n : B_1B_2 \ldots B_nB_1$, and the remaining $\frac{n-1}{3}$ vertices are chosen to be the consecutive apices $A_i\ ;\ 1 \leq i \leq \frac{n-1}{3}$.
Thus, the vertex labeling function f is defined as
$f(B_i) = 0, 1 \leq i \leq n$ and
$f(A_i) = 0, 1 \leq i \leq \frac{n-1}{3}$
This vertex labeling function f,will produce the following edges with label 0 as per the induce edge labeling function f^*.
$f^*(B_iB_{i+1}) = 0\ ; 1 \leq i \leq n-1$
$f^*(B_1B_n) = 0$
$f^*(A_iB_i) = 0\ ;\ 1 \leq i \leq n$
$f^*(A_iC_i) = f^*(A_iD_i) = 0\ ;\ 1 \leq i \leq \frac{n-1}{3}$
Thus, the resultant number of edges getting label 0 is

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$e_0 = (n-1) + (1) + (n) + 2\left(\frac{n-1}{3}\right)$
$= \frac{8n-2}{3}$
which is always greater than $e_{0_R} = 2n$ for all $k = 3, 5, 7, ...$
$\therefore e_0 > e_{0_R}$ for all k.
Thus, J_n does not admit a 3-Cordial labeling in this case either.
Thus, we observe that J_n does not admit a 3-Product Cordial labeling in any case.
Thus, we conclude that a Flower Snark J_n is not a 3-Product Cordial graph.

Theorem 3.3 : A Flower Snark J_n is not a 4-Product Cordial graph.
Proof :
Consider a Flower Snark J_n with $4n$ vertices and $6n$ edges built using n copies of the star graph on 4 vertices, where n is odd. Consider the Vertex set and the Edge set of J_n respectively as
$V = \{A_i, B_i, C_i\ D_i\ ; 1 \le i \le n\}$ and
$E = \{A_iB_i, A_iC_i, A_iD_i\ ; 1 \le i \le n\} \cup \{B_iB_{i+1}, C_iC_{i+1}, D_iD_{i+1}\ ; 1 \le i \le n-1\} \cup \{B_1B_n, C_1D_n, D_1C_n\}$
Thus,
$|V| = n + n + n + n = 4n$ and
$|E| = (n + n + n) + (n - 1 + n - 1 + n - 1) + (1 + 1 + 1) = 3n + (3n - 3) + 3 = 6n$
In order to check for the 4-Product Cordiality, there are four choices from {0, 1, 2, 3} for the labels of the vertices and the edges of the chosen graph. We choose the minimum number of vertices to assign label 0, in order to minimize the number of edges getting label 0, whose probability is 3 in 4 possible outcomes as product of labels 0 and 1. Moreover, these vertices are chosen in maximum adjacency for the same reason.
Thus, $v_0 = \left\lfloor\frac{4n}{4}\right\rfloor = n$ --- (1)
Also, the required number of edges getting label 0 is
$e_{0_R} = \frac{6n}{4} = \frac{3n}{2}$ --- (2)
In order to assign label 0 to n vertices in the maximum possible adjacency, we choose the n vertices of the cycle C_n : $B_1B_2 \dots B_nB_1$.
Thus, the vertex labeling function f is defined as
$f(B_i) = 0, 1 \le i \le n$ and
This vertex labeling function f,will produce the following edges with label 0 as per the induce edge labeling function f^*.
$f^*(B_iB_{i+1}) = 0\ ; 1 \le i \le n-1$
$f^*(B_1B_n) = 0$
$f^*(A_iB_i) = 0\ ;\ 1 \le i \le n$
Thus, the resultant number of edges getting label 0 is
$e_0 = (n-1) + (1) + (n) = 2n$
which is always greater than $e_{0_R} = \frac{3n}{2}$ for all n.
$\therefore e_0 > e_{0_R}$ for all n.
Thus, J_n does not admit a 4-Cordial labeling.
Thus, we conclude that a Flower Snark J_n is not a 4-Product Cordial graph.

Theorem 3.4 : A Flower Snark J_k is not a k-Product Cordial graph.
Proof :
Consider a Flower Snark J_k with $4k$ vertices and $6k$ edges built using k copies of the star graph on 4 vertices, where k is odd. Consider the Vertex set and the Edge set of J_k respectively as
$V = \{A_i, B_i, C_i\ D_i\ ; 1 \le i \le k\}$ and
$E = \{A_iB_i,\ A_iC_i, A_iD_i; 1 \le i \le k\} \cup \{B_iB_{i+1}, C_iC_{i+1}, D_iD_{i+1}\ ; 1 \le i \le k-1\} \cup \{B_1B_k, C_1D_k, D_1C_k\}$

Thus, $|V| = 4k$ and $|E| = 6k$
In order to check for the k-Product Cordiality, there are k choices from $\{0, 1, 2, ..., k-1\}$ for the labels of the vertices and the edges of the chosen graph. We choose the minimum number of vertices to assign label 0, in order to minimize the number of edges getting label 0, whose probability is 3 in 4 possible outcomes as product of labels 0 and 1. Moreover, these vertices are chosen in maximum adjacency for the same reason.
Thus, $v_0 = \frac{4k}{k} = 4$ --- (1)
Also, the required number of edges getting label 0 is
$e_{0_R} = \frac{6k}{k} = 6$ --- (2)

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In order to assign label 0 to 4 vertices in the maximum possible adjacency in J_k, we consider the following 2 cases for the order k.

Case 1 : $k = 3$
The Flower Snark J_3 has a cycle $C_3 : B_1B_2B_3B_1$. Thus, the maximum adjacency for assigning the label 0 to 4 vertices is to label the above 3 vertices B_i and any one apex, say A_1 as zero.
Thus, the vertex labeling function f is defined as
$f(B_i) = 0, 1 \leq i \leq 3$ and
$f(A_1) = 0$
This vertex labeling function f will produce the following edges with label 0 as per the induce edge labeling function f^*.
$f^*(B_iB_{i+1}) = 0 ; 1 \leq i \leq 2$
$f^*(B_1B_3) = 0$
$f^*(A_iB_i) = 0 ; \ 1 \leq i \leq 3$
$f^*(A_1C_1) = f^*(A_1D_1) = 0$
Thus, the resultant number of edges getting label 0 is
$e_0 = 2 + 1 + 3 + 2 = 8$
which is greater than $e_{0_R} = 6$ for $k = 3$.
$\therefore e_0 > e_{0_R}$ for $J_k ; k = 3$.
Thus, J_k does not admit a k-Product Cordial labeling, for $k = 3$.

Case 2 : $k > 3$
A Flower Snark $J_k ; k > 3$ has a cycle $C_k : B_1B_2B_{3\ldots}B_kB_1$ of order greater than 4. Thus, the maximum adjacency for assigning the label 0 to 4 vertices is to label the vertices $B_i ; \ 1 \leq i \leq 4$ as zero.
Thus, the vertex labeling function f is defined as
$f(B_i) = 0, 1 \leq i \leq 4$
This vertex labeling function f will produce the following edges with label 0 as per the induce edge labeling function f^*.
$f^*(B_iB_{i+1}) = 0 ; 1 \leq i \leq 4$
$f^*(B_1B_k) = 0$
$f^*(A_iB_i) = 0 ; \ 1 \leq i \leq 4$
Thus, the resultant number of edges getting label 0 is
$e_0 = 4 + 1 + 4 = 9$
which is greater than $e_{0_R} = 6$ for $k = 3$.
$\therefore e_0 > e_{0_R}$ for $J_k ; k > 3$.
Thus, J_k does not admit a k-Product Cordial labeling, for $k > 3$.
Thus, J_k does not admit a k-Product Cordial labeling for all k.
Thus, J_k is not a k-Product Cordial labeling for all k.

CONCLUSION
1. A Flower Snark J_n is Total Edge Product Cordial.
2. A Flower Snark J_n is not 3-Product Cordial.
3. A Flower Snark J_n is not 4-Product Cordial.
4. A Flower Snark J_k is not k-Product Cordial.

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