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Arithmetic Sequential Graceful Labeling of Complete Bipartite Graph with Pendant Edges

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Abstract: Let G be a simple, finite, connected, undirected, non-trivial graph with p vertices and q edges. $V(G)$ be the vertex set and $E(G)$ be the edge set of G. Let $f: V(G) \rightarrow \{a, a+d, a+2d, a+3d, \dots, a+2qd\}$ where a ≥ 0 and $d \geq 1$ is an injective function. If for each edge $uv \in E(G)$, $f^*: E(G) \rightarrow \{d, 2d, 3d, 4d, \dots, qd\}$ defined by $f^*(uv) = |f(u) - f(v)|$ is a bijective function then the function f is called arithmetic sequential graceful labeling. The graph with arithmetic sequential graceful labeling is called arithmetic sequential graceful graph. In this paper, we proved complete bipartite graph with pendant edges are arithmetic sequential graceful graph.

Keywords: Graceful labeling, Arithmetic sequential graceful labeling, Complete bipartite graph

1. Introduction

Graceful labeling is a classical graph theory concept where each vertex of a graph is assigned a unique integer label, ensuring that the absolute differences between labels of adjacent vertices are distinct. In complete bipartite graphs $k_{m,n}$with pendant edges, this labeling technique involves assigning labels to the vertices such that these differences remain distinct despite the additional complexity introduced by pendant edges. A complete bipartite graph consists of two disjoint vertex sets, with every vertex in one set connected to all vertices in the other set. Pendant edges connect a vertex of degree one to the graph, adding further constraints to the labeling process. In this paper, we proved complete bipartite graph with pendant edges are arithmetic sequential graceful graph.

2. Definition

Defintion 2.1: A graph obtained by attaching l pendant edges to the vertex u_1 of the complete bipartite graph $K_{m,n}$ with the vertex set $\{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\}$ is denoted by $k_{m,n} \odot u_1(l)$.
Defintion 2.2: A graph obtained by attaching l_1 pendant edges to the vertex u_1 and attaching l_2 pendant edges to the vertex u_m of the complete bipartite graph $K_{m,n}$ with the vertex set $\{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\}$ is denoted by $k_{m,n} \odot u_{1,m}(l_1, l_2)$.
Defintion 2.3: A graph obtained by attaching l_1 , l_2 and l_3 pendant edges to the vertex u_1 , u_m and v_1 of the complete bipartite graph $K_{m,n}$ with the vertex set $\{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\}$ is denoted by $k_{m,n} \odot u_{1,m}v_1(l_1, l_2, l_3)$.
Defintion 2.4: A graph obtained by attaching l_1 , l_2 , l_3 and l_4 pendant edges to the vertex u_1 , u_m , $v_1$and v_n of the complete bipartite graph $K_{m,n}$ with the vertex set $\{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\}$ is denoted by $k_{m,n} \odot u_{1,m}v_{1,n}(l_1, l_2, l_3, l_4)$.

3. Main Result

Theorem 3.1: The graph $k_{m,n} \odot u_1(l)$ admits Arithmetic sequential graceful labeling.
Proof: Let G be a $k_{m,n} \odot u_1(l)$ graph.
$V(G) = \{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\} \cup \{x_k: 1 \leq k \leq l\}$ and
$E(G) = \{u_i\, v_j: 1 \leq i \leq m;\ 1 \leq j \leq n\} \cup \{u_1\, x_k: 1 \leq k \leq l\}$.
$|V(G)| = m + n + l$
$|E(G)| = m\,n + l$
We define a function $f : V(G) \rightarrow \{a, a+d, a+2d, a+3d, \dots, 2(a+qd)\}$
The vertex labeling is as follows:
$f(u_i) = a + [i-1]d \quad 1 \leq i \leq m$
$f(v_j) = a + [m\,j]d \quad 1 \leq j \leq n$
$f(x_k) = a + [m\,n + k]d \quad 1 \leq k \leq l$
By above labeling pattern, we observed that function $f : V(G) \rightarrow \{a, a+d, a+2d, a+3d, \dots, a+2qd\}$ is $1-1$.
From the induced function $f^*: E(G) \rightarrow \{d, 2d, 3d, 4d, \dots, qd\}$, we get the edge labels as follows.

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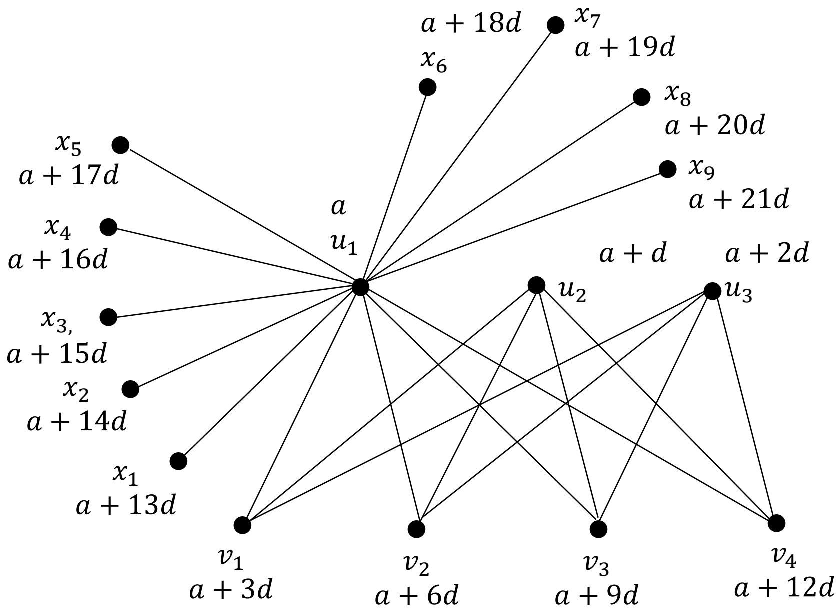
Table 1: Edge labels of the graph $k_{m,n} \odot u_1(l)$

f^*	Edge Labels	Value of i, j and k
$\lvert f(u_i) - f(v_j) \rvert$	$[1 - i + m\,j]d$	$1 \le i \le m, 1 \le j \le n$
$\lvert f(u_1) - f(x_k) \rvert$	$[m\,n + k]d$	$1 \le k \le l$

From the above table 1, we observed that $f^*: E(G) \to \{d, 2d, 3d, 4d, \ldots, qd\}$ defined by $f^*(uv) = |f(u) - f(v)|$ is a bijective. Hence, f is arithmetic sequential graceful labeling and the graph $k_{m,n} \odot u_1(l)$ is arithmetic sequential graceful graph.

Example 3.1.1: Arithmetic sequential graceful labeling of the graph $k_{3,4} \odot u_1(9)$



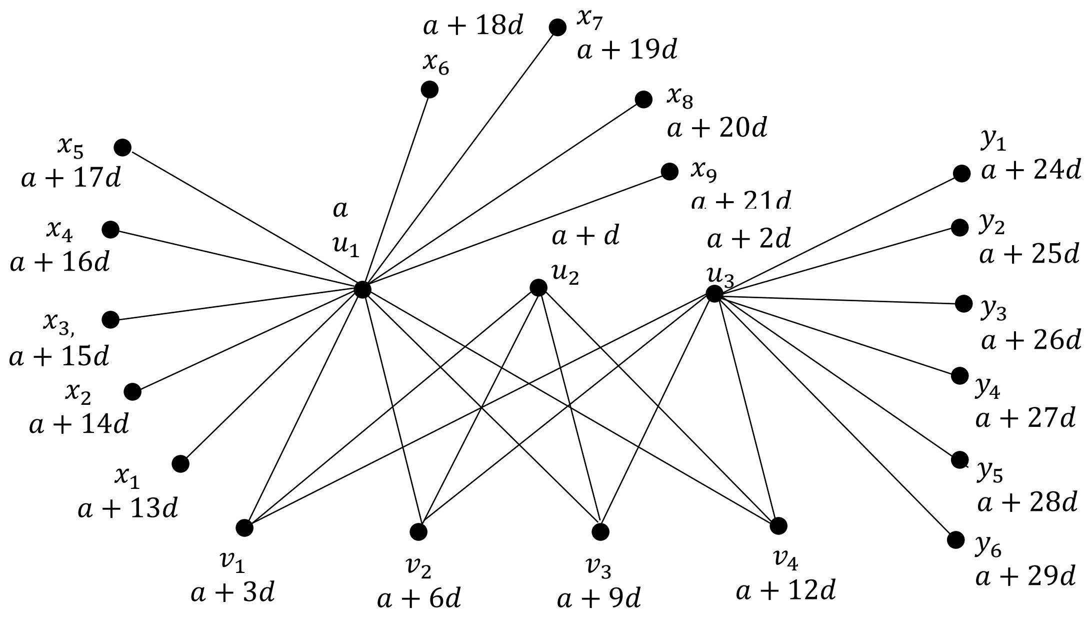
Theorem 3.2: The graph $k_{m,n} \odot u_{1,m}(l_1, l_2)$ admits Arithmetic sequential graceful labeling.
Proof: Let G be a $k_{m,n} \odot u_{1,m}(l_1, l_2)$ graph.
$V(G) = \{u_i: 1 \le i \le m\} \cup \{v_j: 1 \le j \le n\} \cup \{x_k: 1 \le k \le l_1\} \cup \{y_k: 1 \le k \le l_2\}$ and
$E(G) = \{u_i\, v_j: 1 \le i \le m;\ 1 \le j \le n\} \cup \{u_1\, x_k: 1 \le k \le l_1\} \cup \{u_m\, y_k: 1 \le k \le l_2\}$.
$|V(G)| = m + n + l_1 + l_2$
$|E(G)| = m\,n + l_1 + l_2$
We define a function $f: V(G) \to \{a, a+d, a+2d, a+3d, \ldots, a+2qd\}$
The vertex labeling is as follows:
$f(u_i) = a + [i - 1]d$ $\quad 1 \le i \le m$
$f(v_j) = a + [m\,j]d$ $\quad 1 \le j \le n$
$f(x_k) = a + [m\,n + k]d$ $\quad 1 \le k \le l_1$
$f(y_k) = a + [m\,n + l_1 + m - 1 + k]d$ $\quad 1 \le k \le l_2$
By above labeling pattern, we observed that $f: V(G) \to \{a, a+d, a+2d, a+3d, \ldots, a+2qd\}$ is $1 - 1$.
From the induced function $f^*: E(G) \to \{d, 2d, 3d, 4d, \ldots, qd\}$, we get the edge labels as follows.

Table 2: Edge labels of the graph $k_{m,n} \odot u_{1,m}(l_1, l_2)$

f^*	Edge Labels	Value of i, j and k
$\lvert f(u_i) - f(v_j) \rvert$	$[1 - i + m\,j]d$	$1 \le i \le m, 1 \le j \le n$
$\lvert f(u_1) - f(x_k) \rvert$	$[m\,n + k]d$	$1 \le k \le l_1$
$\lvert f(u_m) - f(y_k) \rvert$	$[m\,n + l_1 + k]d$	$1 \le k \le l_2$

From the above table 2, we observed that $f^*: E(G) \to \{d, 2d, 3d, 4d, \dots, qd\}$ defined by $f^*(uv) = |f(u) - f(v)|$ is a bijective. Hence, f is arithmetic sequential graceful labeling and the graph $k_{m,n} \odot u_{1,m}(l_1, l_2)$ is arithmetic sequential graceful graph.

Example 3.2.1: Arithmetic sequential graceful labeling of the graph $k_{3,4} \odot u_{1,3}(9, 6)$



Theorem 3.3: The graph $k_{m,n} \odot u_{1,m}v_1(l_1, l_2, l_3)$ admits Arithmetic sequential graceful labeling.

Proof: Let G be a $k_{m,n} \odot u_{1,m}v_1(l_1, l_2, l_3)$ graph.
$V(G) = \{u_i: 1 \le i \le m\} \cup \{v_j: 1 \le j \le n\} \cup \{x_k: 1 \le k \le l_1\} \cup \{y_k: 1 \le k \le l_2\}$
$\cup \{z_k: 1 \le k \le l_3\}$ and
$E(G) = \{u_i\, v_j: 1 \le i \le m;\ 1 \le j \le n\} \cup \{u_1\, x_k: 1 \le k \le l_1\} \cup \{u_m\, y_k: 1 \le k \le l_2\}$
$\cup \{v_1\, y_k: 1 \le k \le l_3\}$..
$|V(G)| = m + n + l_1 + l_2 + l_3$
$|E(G)| = m\,n + l_1 + l_2 + l_3$

We define a function $f : V(G) \to \{a, a + d, a + 2d, a + 3d, \dots, a + 2qd\}$

The vertex labeling is as follows:
$f(u_i) = a + [i - 1]d$ $\quad 1 \le i \le m$
$f(v_j) = a + [m\,j]d$ $\quad 1 \le j \le n$
$f(x_k) = a + [m\,n + k]d$ $\quad 1 \le k \le l_1$
$f(y_k) = a + [m\,n + l_1 + m - 1 + k]d$ $\quad 1 \le k \le l_2$
$f(z_k) = a + [m\,n + l_1 + l_2 + m + k]d$ $\quad 1 \le k \le l_3$

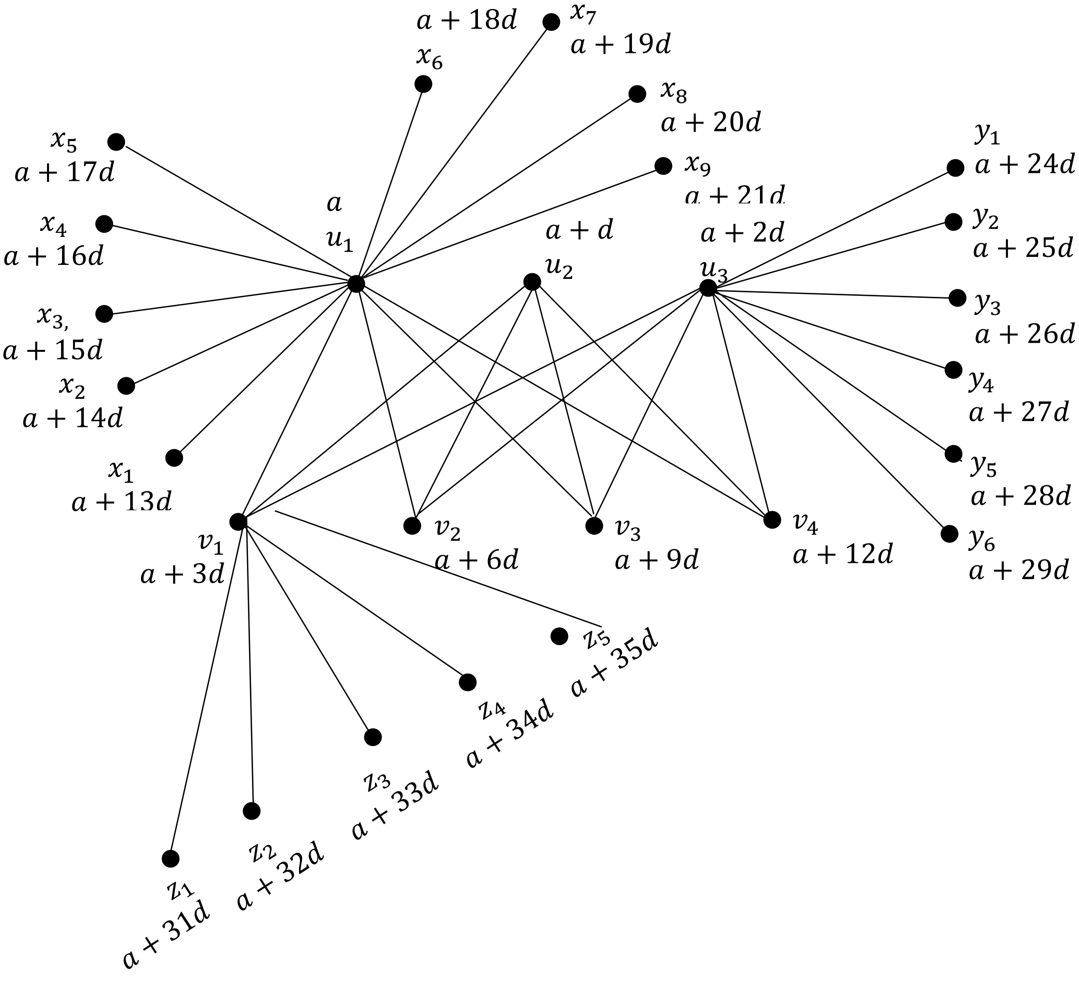
By above labeling pattern, we observed that function $f : V(G) \to \{a, a + d, a + 2d, a + 3d, \dots, a + 2qd\}$ is $1 - 1$.
From the induced function $f^*: E(G) \to \{d, 2d, 3d, 4d, \dots, qd\}$, we get the edge labels as follows.

Table 3: Edge labels of the graph $k_{m,n} \odot u_{1,m}v_1(l_1, l_2, l_3)$.

f^*	Edge Labels	Value of i, j and k
$\lvert f(u_i) - f(v_j) \rvert$	$[1 - i + m\, j]d$	$1 \le i \le m, 1 \le j \le n$
$\lvert f(u_1) - f(x_k) \rvert$	$[m\, n + k]d$	$1 \le k \le l_1$
$\lvert f(u_m) - f(y_k) \rvert$	$[m\, n + l_1 + k]d$	$1 \le k \le l_2$
$\lvert f(v_1) - f(z_k) \rvert$	$[m\, n + l_1 + l_2 + k]d$	$1 \le k \le l_3$

From the above table 3, we observed that $f^*: E(G) \to \{d, 2d, 3d, 4d, \dots, qd\}$ defined by $f^*(uv) = |f(u) - f(v)|$ is a bijective. Hence, f is arithmetic sequential graceful labeling and the graph $k_{m,n} \odot u_{1,m}v_1(l_1, l_2, l_3)$ is arithmetic sequential graceful graph.

Example 3.3.1: Arithmetic sequential graceful labeling of the graph $k_{3,4} \odot u_{1,3}v_1(9, 6, 5)$.



Theorem 3.4: The graph $k_{m,n} \odot u_{1,m}v_{1,n}(l_1, l_2, l_3, l_4)$ admits Arithmetic sequential graceful labeling.
Proof: Let G be a $k_{m,n} \odot u_{1,m}v_{1,n}(l_1, l_2, l_3, l_4)$ graph.

$V(G) = \{u_i: 1 \le i \le m\} \cup \{v_j: 1 \le j \le n\} \cup \{x_k: 1 \le k \le l_1\} \cup \{y_k: 1 \le k \le l_2\}$
$\cup \{z_k: 1 \le k \le l_3\} \cup \{w_k: 1 \le k \le l_4\}$ and
$E(G) = \{u_i\, v_j: 1 \le i \le m;\ 1 \le j \le n\} \cup \{u_1\, x_k: 1 \le k \le l_1\} \cup \{u_m\, y_k: 1 \le k \le l_2\}$
$\cup \{v_1\, y_k: 1 \le k \le l_3\} \cup \{v_n\, w_k: 1 \le k \le l_4\}$.

$|V(G)| = m + n + l_1 + l_2 + l_3$
$|E(G)| = m\, n + l_1 + l_2 + l_3$
We define a function $f : V(G) \to \{a, a + d, a + 2d, a + 3d, \dots, a + 2qd\}$
The vertex labeling is as follows:
$f(u_i) = a + [i - 1]d$ $\qquad 1 \le i \le m$

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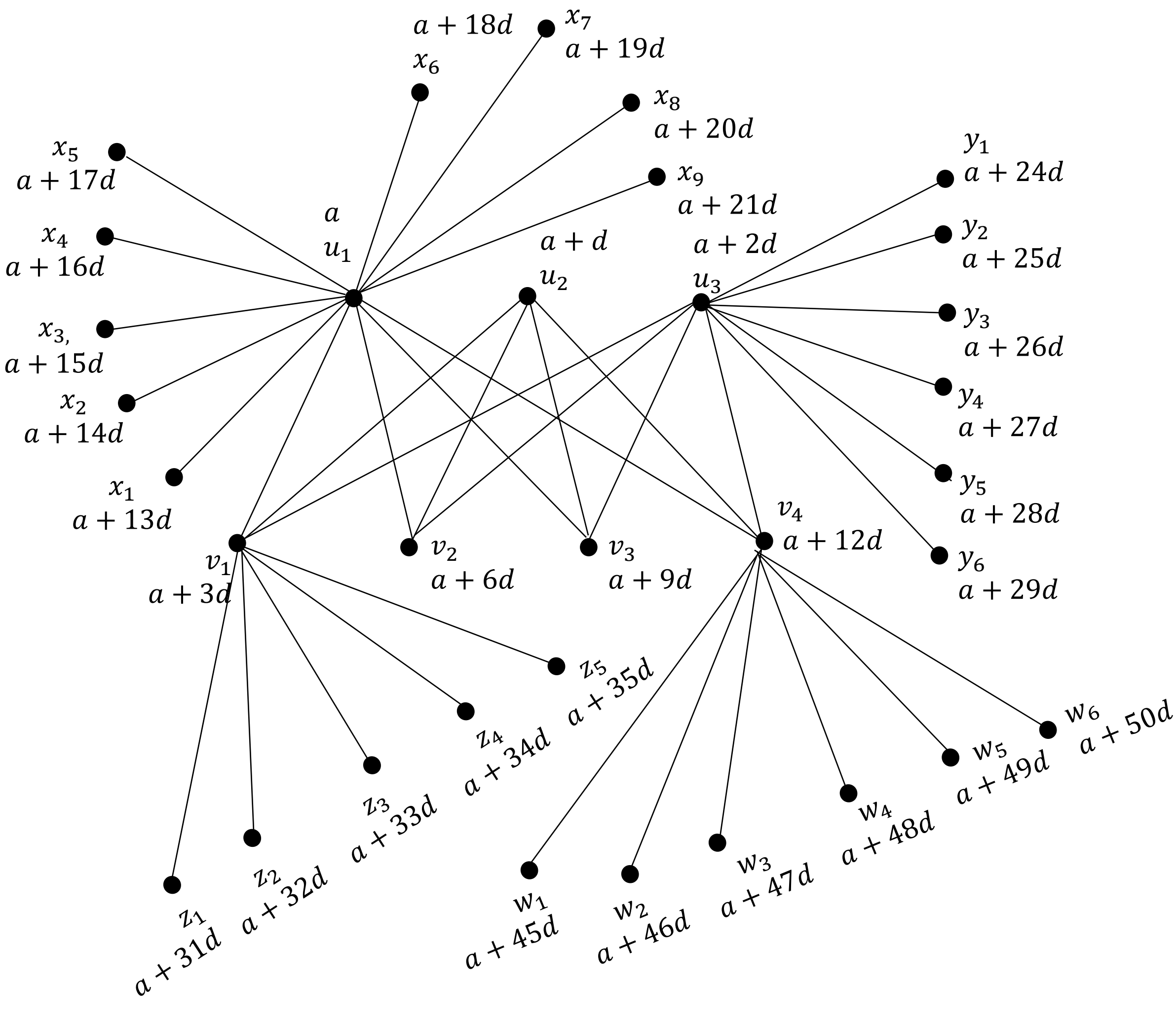
$f(v_j) = a + [m\,j]d \qquad 1 \le j \le n$
$f(x_k) = a + [m\,n + k]d \qquad 1 \le k \le l_1$
$f(y_k) = a + [m\,n + l_1 + m - 1 + k]d \qquad 1 \le k \le l_2$
$f(z_k) = a + [m\,n + l_1 + l_2 + m + k]d \qquad 1 \le k \le l_3$
$f(w_k) = a + [2m\,n + l_1 + l_2 + l_3 + k]d \qquad 1 \le k \le l_4$
By above labeling pattern, we observed that function $f : V(G) \to \{a, a+d, a+2d, a+3d, \dots, a+2qd\}$ is $1-1$.
From the induced function $f^*: E(G) \to \{d, 2d, 3d, 4d, \dots, qd\}$, we get the edge labels as follows.

Table 4: Edge labels of the graph $k_{m,n} \odot u_{1,m}v_{1,n}(l_1, l_2, l_3, l_4)$.

f^*	Edge Labels	Value of i, j and k
$\lvert f(u_i) - f(v_j)\rvert$	$[1 - i + m\,j]d$	$1 \le i \le m, 1 \le j \le n$
$\lvert f(u_1) - f(x_k)\rvert$	$[m\,n + k]d$	$1 \le k \le l_1$
$\lvert f(u_m) - f(y_k)\rvert$	$[m\,n + l_1 + k]d$	$1 \le k \le l_2$
$\lvert f(v_1) - f(z_k)\rvert$	$[m\,n + l_1 + l_2 + k]d$	$1 \le k \le l_3$
$\lvert f(v_n) - f(w_k)\rvert$	$[m\,n + l_1 + l_2 + l_3 + k]d$	$1 \le k \le l_4$

From the above table 4, we observed that $f^*: E(G) \to \{d, 2d, 3d, 4d, \dots, qd\}$ defined by $f^*(uv) = |f(u) - f(v)|$ is a bijective. Hence, f is arithmetic sequential graceful labeling and the graph $k_{m,n} \odot u_{1,m}v_{1,n}(l_1, l_2, l_3, l_4)$ is arithmetic sequential graceful graph.

Example 3.4.1: Arithmetic sequential graceful labeling of the graph $k_{3,4} \odot u_{1,3}v_{1,4}(9, 6,5, 6)$.



4. Conclusion

In this Paper, we proved some complete bipartite graph with pendant edges are arithmetic sequential graceful graph. Labeling pattern is demonstrated by means of illustrations, which provide better understanding of derived results. Analysing arithmetic sequential graceful on other families of graph are our future work.

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