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Cardinal labeling of H-graphs

A. Punitha Tharani[1], S. Pratiksha[2*]

[1]Associate Professor, Department of Mathematics, St.Mary's College (Autonomous), Thoothukudi-628 001, Tamil Nadu, India, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627 012, Tamil Nadu, India.
[1]punitha_tharani@yahoo.co.in
[2*] Research Scholar (Register number: 21212212092004), Department of Mathematics, St.Mary's College (Autonomous), Thoothukudi-628 001, Tamil Nadu, India, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627 012, Tamil Nadu, India., pratikshaprabhu32@gmail.com

ABSTRACT
Let $G(V,E)$ be a simple connected undirected graph. Let $|V| = p, |E| = q$ and let $C = \{1,2,\dots,p+q\}$. G is said to admit cardinal labeling if the one-to-one function f from the vertex set V into the set C generates a one-to-one edge function f^* onto the set $C \setminus f(V)$ defined by $|f(u) - f(v)| = f^*(uv)\ \forall\ uv \in E$. A graph G is said to be cardinal if it admits cardinal labeling. In this paper we have discussed cardinal labeling for H-graph H_n, subdivided H-graph $S(H_n)$, corona of H-graph $H_n \odot K_1$, $H_n \odot mK_1$, and H-cracker graph $H_{n,k}$.

Keypoints: cardinal, graph subdivision, graph corona, H-graph, H-cracker graph

AMS subject classification MSC (2010) No: 05C78

1 INTRODUCTION

Motivated by magic and graceful labelings [1], our work is based on a new type of labeling wherein we optimally assign unique non-repetitive labels to the vertices and the edges of a graph, ranging from 1 up to the sum of the order and size of the graph, where the edge labels are simply the absolute difference of the vertex labels. We call this labeling as cardinal labeling and any graph admitting this labeling can rightly be called cardinal. All the graphs that we take into consideration are simple, connected and undirected. This work discusses the labeling for graphs like H-graph, subdivided H-graph $S(H_n)$, corona *of* H-graph $H_n \odot K_1$, $H_n \odot mK_1$ and H-cracker graph $H_{n,k}$.

Definition 1.1
Let $G(V,E)$ be a simple connected undirected graph. Let $|V| = p, |E| = q$ and let $C = \{1,2,\dots,p+q\}$. G is said to admit cardinal labeling if the one-to-one function f from the vertex set V into the set C generates a one-to-one edge function f^* onto the set $C \setminus f(V)$ defined by $|f(u) - f(v)| = f^*(uv)\ \forall\ uv \in E$. A graph G is said to be cardinal if it admits cardinal labeling.

Definition 1.2 [2]
The H-graph is the graph obtained from two copies of path P_n with vertices $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ by joining the vertices $u_{\frac{n+1}{2}}$ and $v_{\frac{n+1}{2}}$ if n is odd and $u_{\frac{n}{2}+1}$ and $v_{\frac{n}{2}}$ if n is even. We denote H-graph by H_n.

Definition 1.3 [3]
Let G_1, G_2 respectively be $(p_1, q_1), (p_2, q_2)$ graphs. The corona of G_1 with G_2 is the graph $G_1 \odot G_2$ obtained by taking one copy of G_1, p_1 copies of G_2 and joining the i'th vertex of G_1 by an edge to every vertex in the i'th copy of G_2 where $1 \leq i \leq p_1$.

Definition 1.4 [4]
The graph $H_n \odot K_1$ is obtained by adding a pendant edge to each vertex of H-graph H_n.

Definition 1.5 [5]
The graph $H_n \odot mK_1$ is a graph obtained from H-graph H_n by attaching m pendant vertices at each i^{th} vertex on the two paths on n vertices for $1 \leq i \leq n$.

Definition 1.6 [3]
If $e = uv$ is an edge of G then e is said to be subdivided when it is replaced by the edges uw and wv, where w is the vertex that subdivides edge uv. The graph obtained by subdividing each edge of a graph G is called the subdivision graph of G and is denoted by $S(G)$.

Definition 1.7
Let G be a graph obtained by the concatenation of $2n$ stars $K_{1,k}$ to H-graph H_n by attaching the central vertices of each $K_{1,k}$ to each i^{th} vertex on the two paths on n vertices of H_n for $1 \leq i \leq n$. We call this graph as H-cracker graph and we denote this graph as $H_{n,k}$.

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2 MAIN RESULTS

Theorem 2.1

H-graph H_n is cardinal.

Proof

The H-graph is the graph obtained from two copies of path P_n with vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ by joining the vertices $u_{\frac{n+1}{2}}$ and $v_{\frac{n+1}{2}}$ if n is odd and $u_{\frac{n}{2}+1}$ and $v_{\frac{n}{2}}$ if n is even. It has $2n$ vertices and $2n-1$ edges, so C becomes $C = \{1,2,\ldots,4n-1\}$. We define the vertex function $f: V \to C$ for the graph as follows.

$$f(u_i) = \begin{cases} 4n-i & \text{when } i \text{ is odd}, \quad \forall\, n \\ i-1 & \text{when } i \text{ is even}, \quad \forall\, n \end{cases}$$

Case (i) When n is odd, $f(v_i) = \begin{cases} n+i-1 & \text{when } i \text{ is odd} \\ 3n-i & \text{when } i \text{ is even} \end{cases}$

Case (ii) When n is even, $f(v_i) = \begin{cases} 3n-i & \text{when } i \text{ is odd} \\ n+i-1 & \text{when } i \text{ is even} \end{cases}$

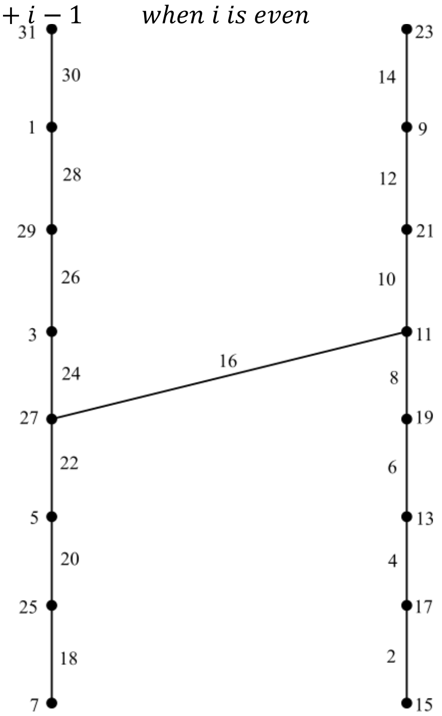

Figure 1: H-graph H_8

These functions assign members of C uniquely to the vertices of the graph. The edge labels are assigned under the induced functions $f^*: E \to C \backslash f(V)$; i.e., by finding the absolute difference of the corresponding vertex labels.

$$f^*(u_i u_{i+1}) = |4n - i - ((i+1)-1)| = 2(2n-i), \quad \text{when } i \text{ is odd}, \quad \forall\, n$$
$$f^*(u_i u_{i+1}) = \left|i - 1 - \left(4n - (i+1)\right)\right| = 2(2n-i), \quad \text{when } i \text{ is even}, \quad \forall\, n$$

Case (i) When n is odd

$$f^*(v_i v_{i+1}) = \begin{cases} |n+i-1-(3n-(i+1))| = |-2(n-i)| = 2(n-i), & \text{when } i \text{ is odd} \\ |3n-i-(n+(i+1)-1)| = 2(n-i), & \text{when } i \text{ is even} \end{cases}$$

$$f^*\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\right) = \begin{cases} \left|4n - \left(\frac{n+1}{2}\right) - \left(n + \left(\frac{n+1}{2}\right) - 1\right)\right| = 2n, & \text{when } \frac{n+1}{2} \text{ is odd} \\ \left|\frac{n+1}{2} - 1 - \left(3n - \left(\frac{n+1}{2}\right)\right)\right| = |-2n| = 2n, & \text{when } \frac{n+1}{2} \text{ is even} \end{cases}$$

Case (ii) When n is even

$$f^*(v_i v_{i+1}) = \begin{cases} |3n-i-(n+(i+1)-1)| = 2(n-i), & \text{when } i \text{ is odd} \\ \left|n+i-1-\left(3n-(i+1)\right)\right| = |-2(n-i)| = 2(n-i), & \text{when } i \text{ is even} \end{cases}$$

$$f^*\left(u_{\frac{n}{2}+1} v_{\frac{n}{2}}\right) = \begin{cases} \left|\left(\frac{n}{2}+1\right) - 1 - \left(3n - \frac{n}{2}\right)\right| = |-2n| = 2n, & \text{when } \frac{n}{2} \text{ is odd} \\ \left|4n - \left(\frac{n}{2}+1\right) - \left(n + \frac{n}{2} - 1\right)\right| = 2n, & \text{when } \frac{n}{2} \text{ is even} \end{cases}$$

On doing so, it could be seen that all the edges receive members of C other than the ones that were given to the vertices; i.e. C is covered by the labels of the vertices and edges disjointly. Hence H-graph admits cardinal labeling.

Theorem 2.2
Subdivided H-graph $S(H_n)$ satisfies cardinal labeling.
Proof
Let $u_1, u_2, \dots, u_n$ *and* $v_1, v_2, \dots, v_n$ be the vertices of the two paths of length $n-1$ of H_n. Each edge u_iu_{i+1} is subdivided by the vertex u_i', $1 \le i \le n-1$, and each edge v_iv_{i+1} is subdivided by the vertex v_i', $1 \le i \le n-1$. The edge $u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}$ is divided by a vertex w when n is odd and the edge $u_{\frac{n}{2}+1}v_{\frac{n}{2}}$ is divided by a vertex w when n is even. The graph $S(H_n)$ has $4n-1$ vertices and $4n-2$ edges. We have $C = \{1,2,\dots,8n-3\}$. Define $f: V \to C$ as follows.

$$f(u_i) = 2i-1, \qquad i = 1,2,\dots,n$$

$$f(u_i') = (8n-1) - 2i, \qquad i = 1,2,\dots,n-1$$

$$f(w) = \begin{cases} 5n & when\ n\ is\ odd \\ 5n+1 & when\ n\ is\ even \end{cases}$$

$$f(v_i) = (2n-1) + 2i, \qquad i = 1,2,\dots,n$$

$$f(v_i') = \begin{cases} (6n+1) - 2i,\ i = 1,2,\dots,\frac{n-1}{2} & \\ (6n-1) - 2i,\ i = \frac{n+1}{2},\dots,n-1 & when\ n\ is\ odd \\ (6n+1) - 2i,\ i = 1,2,\dots,\frac{n-2}{2} & \\ (6n-1) - 2i,\ i = \frac{n}{2},\dots,n-1 & when\ n\ is\ even \end{cases}$$

For the vertex labeling f, the induced edge labeling $f^*: E \to C \setminus f(V)$ is given as follows.

$$f^*(u_iu_i') = \left|2i - 1 - \left((8n-1) - 2i\right)\right| = |-8n + 4i| = 4(2n-i),$$
$$i = 1,2,\dots,n-1, \qquad \forall\, n$$

$$f^*(u_i'u_{i+1}) = |8n - 1 - 2i - (2(i+1) - 1)| = 4(2n-i) - 2,$$
$$i = 1,2,\dots,n-1, \qquad \forall\, n$$

Case (i) When n is odd

$$f^*\left(u_{\frac{n+1}{2}}w\right) = \left|2\left(\frac{n+1}{2}\right) - 1 - 5n\right| = 4n$$

$$f^*\left(wv_{\frac{n+1}{2}}\right) = \left|5n - \left((2n-1) + 2\left(\frac{n+1}{2}\right)\right)\right| = 2n$$

$$f^*(v_iv_i') = |2n - 1 + 2i - (6n + 1 - 2i)| = 4(n-i) + 2, \qquad i = 1,2,\dots,\frac{n-1}{2}$$

$$f^*(v_i'v_{i+1}) = \left|6n + 1 - 2i - \left(2n - 1 + 2(i+1)\right)\right| = 4(n-i), \qquad i = 1,2,\dots,\frac{n-1}{2}$$

$$f^*(v_iv_i') = \left|2n - 1 + 2i - \left((6n-1) - 2i\right)\right| = 4(n-i), \qquad i = \frac{n+1}{2},\dots,n-1$$

$$f^*(v_i'v_{i+1}) = \left|6n - 1 - 2i - \left(2n - 1 + 2(i+1)\right)\right| = 4(n-i) - 2, i = \frac{n+1}{2},\dots,n-1$$

Case (ii) When n is even

$$f^*\left(u_{\frac{n}{2}+1}w\right) = \left|2\left(\frac{n}{2} + 1\right) - 1 - (5n+1)\right| = 4n$$

$$f^*\left(wv_{\frac{n}{2}}\right) = \left|5n + 1 - \left((2n-1) + 2\left(\frac{n}{2}\right)\right)\right| = 2(n+1)$$

$$f^*(v_iv_i') = |2n - 1 + 2i - (6n + 1 - 2i)| = 4(n-i) + 2, \qquad i = 1,2,\dots,\frac{n-2}{2}$$

$$f^*(v_i'v_{i+1}) = \left|6n + 1 - 2i - \left(2n - 1 + 2(i+1)\right)\right| = 4(n-i), \qquad i = 1,2,\dots,\frac{n-2}{2}$$

$$f^*(v_iv_i') = \left|2n - 1 + 2i - \left((6n-1) - 2i\right)\right| = 4(n-i), \qquad i = \frac{n}{2},\dots,n-1$$

$$f^*(v_i'v_{i+1}) = \left|6n - 1 - 2i - \left(2n - 1 + 2(i+1)\right)\right| = 4(n-i) - 2, i = \frac{n}{2},\dots,n-1$$

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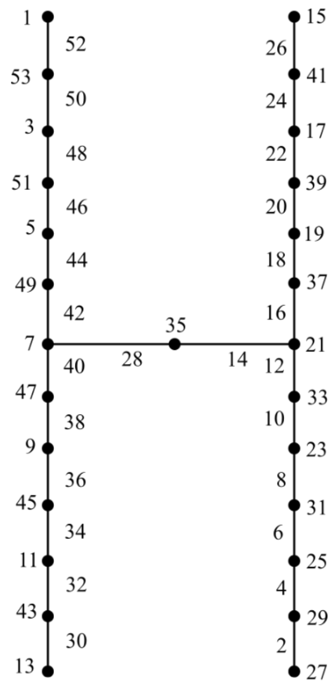

Figure 2: $\boldsymbol{S(H_7)}$

Now, both the vertex set and edge set are distinctly labelled with the members of C which defends our definition. Hence subdivided H-graph is cardinal.

Theorem 2.3
$H_n \odot K_1$ admits cardinal labeling.
Proof
Let $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ be the two path vertices of H_n and $u_1', u_2', \dots, u_n'$ and $v_1', v_2', \dots, v_n'$ be the pendant vertices adjacent to $u_i s$ and $v_i s$ respectively $(i = 1,2, \dots, n)$. $H_n \odot K_1$ has $4n$ vertices and $4n - 1$ edges and so $C = \{1,2, \dots, 8n - 1\}$. Define $f: V \to C$ by

$$f(u_i) = \begin{cases} 8n + 1 - 2i & when\ i\ is\ odd \\ 2i - 1 & when\ i\ is\ even \end{cases}, \forall\ n$$

$$f(u_i') = \begin{cases} 2i - 1 & when\ i\ is\ odd \\ 8n + 1 - 2i & when\ i\ is\ even \end{cases}, \forall\ n$$

Case (i) When n is odd

$$f(v_i) = \begin{cases} 2n - 1 + 2i & when\ i\ is\ odd \\ 6n + 1 - 2i & when\ i\ is\ even \end{cases}$$

$$f(v_i') = \begin{cases} 6n + 1 - 2i & when\ i\ is\ odd \\ 2n - 1 + 2i & when\ i\ is\ even \end{cases}$$

Case (ii) When n is even

$$f(v_i) = \begin{cases} 6n + 1 - 2i & when\ i\ is\ odd \\ 2n - 1 + 2i & when\ i\ is\ even \end{cases}$$

$$f(v_i') = \begin{cases} 2n - 1 + 2i & when\ i\ is\ odd \\ 6n + 1 - 2i & when\ i\ is\ even \end{cases}$$

The induced edge function f^* onto the set $C \backslash f(V)$ is given below.

$$f^*(u_i u_{i+1}) = |(8n + 1) - 2i - (2(i + 1) - 1)| = 4(2n - i), \quad i\ is\ odd, \quad \forall\ n$$

$$f^*(u_i u_{i+1}) = \left|(2i - 1) - \left((8n + 1) - 2(i + 1)\right)\right| = 4(2n - i), \quad i\ is\ odd, \quad \forall\ n$$

$$f^*(u_i u_i') = |(8n + 1 - 2i) - (2i - 1)| = 4(2n - i) + 2, i\ is\ odd, \forall\ n$$

$$f^*(u_i u_i') = |(2i - 1) - (8n + 1 - 2i)| = |8n - 4i + 2| = 4(2n - i) + 2, i\ is\ even, \forall\ n$$

Case (i) When n is odd

$$f^*\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\right) = \left|8n + 1 - 2\left(\frac{n+1}{2}\right) - \left((2n - 1) + 2\left(\frac{n+1}{2}\right)\right)\right| = 4n,\ when\ \frac{n+1}{2}\ is\ odd$$

$$f^*\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\right) = \left|2\left(\frac{n+1}{2}\right) - 1 - \left(6\left(\frac{n+1}{2}\right) + 1\right)\right| = |-4n| = 4n,\ when\ \frac{n+1}{2}\ is\ even$$

$$f^*(v_i v_{i+1}) = \left|2n - 1 + 2i - \left(6n + 1 - 2(i + 1)\right)\right| = 4(n - i), \quad i\ is\ odd$$

$$f^*(v_i v_{i+1}) = \left|6n + 1 - 2i - \left(2n - 1 + 2(i + 1)\right)\right| = 4(n - i), \quad i\ is\ even$$

$$f^*(v_i v_i') = |2n - 1 + 2i - (6n + 1 - 2i)| = |-4n - 2 + 4i| = 4(n - i) + 2, \quad i\ is\ odd$$

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$$f^*(v_i v_i') = |6n+1-2i-(2n-1+2i)| = 4(n-i)+2, \qquad i \text{ is even}$$

Case (ii) When n is even

$$f^*(v_i v_{i+1}) = \left|6n+1-2i-\left(2n-1+2(i+1)\right)\right| = 4(n-i), \qquad i \text{ is odd}$$
$$f^*(v_i v_{i+1}) = \left|2n-1+2i-\left(6n+1-2(i+1)\right)\right| = 4(n-i), \qquad i \text{ is even}$$
$$f^*(v_i v_i') = |6n+1-2i-(2n-1+2i)| = 4(n-i)+2, \qquad i \text{ is odd}$$
$$f^*(v_i v_i') = |2n-1+2i-(6n+1-2i)\ | = 4(n-i)+2, \qquad i \text{ is even}$$
$$f^*\left(u_{\frac{n}{2}+1} v_{\frac{n}{2}}\right) = \left|2\left(\frac{n}{2}+1\right)-1-\left(6n+1-2\left(\frac{n}{2}\right)\right)\right| = |-4n| = 4n, when\ \frac{n}{2}\ \text{ is odd}$$
$$f^*\left(u_{\frac{n}{2}+1} v_{\frac{n}{2}}\right) = \left|8n+1-2\left(\frac{n+1}{2}\right)-\left(2n-1+2\left(\frac{n}{2}\right)\right)\right| = 4n, when\ \frac{n}{2}\ \text{is even}$$

The resultant edge labels obtained by the absolute difference of the incident vertex labels are all distinct in C. Also all the vertices of $H_n \odot K_1$ have distinct labels of C that were not given to $\{E\}$. The resultant labelled corona graph of H_n is hence cardinal.

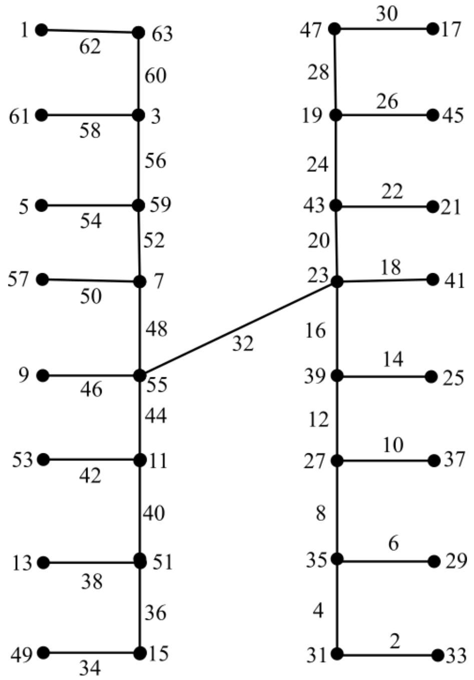

Figure 3: ***$H_8 \odot K_1$***

Theorem 2.4
$H_n \odot mK_1$, $m \geq 2$ is cardinal.
Proof
Let $u_i s$ and $v_i s$, $1 \leq i \leq n$, denote the vertices of the two paths on n vertices and $u_{ij}s$ *and* $v_{ij}s$, $1 \leq j \leq m$, denote the pendant vertices attached to each vertex of the two paths of H_n. The graph has $2n(m+1)$ vertices and $2n(m+1)-1$ edges. We have $C = \{1,2,\dots,4n(m+1)-1\}$. We begin labeling the vertices of $H_n \odot mK_1$ under the function $f: V \to C$ defined as follows.

$$f(u_i) = \begin{cases} i+m(i-1) & when\ i\ is\ odd, \quad \forall\, n \\ (4n-i)(m+1)+1 & when\ i\ is\ even, \quad \forall\, n \end{cases}$$
$$f(u_{ij}) = \begin{cases} (m+1)(4n+1-i)-(2j-1) & when\ i\ is\ odd, \quad j=1,2,\dots,m, \quad \forall\, n \\ (m+1)(i-2)+(2j+1) & when\ i\ is\ even, \quad j=1,2,\dots,m, \quad \forall\, n \end{cases}$$

Case (i) When n is odd

$$f(v_i) = \begin{cases} (m+1)(3n-i)+1 & when\ i\ is\ odd \\ (m+1)(n-1+i)+1 & when\ i\ is\ even \end{cases}$$
$$f(v_{ij}) = \begin{cases} (m+1)(n-2+i)+(2j+1) & when\ i\ is\ odd, \quad j=1,2,\dots,m \\ (m+1)(3n+1-i)-(2j-1) & when\ i\ is\ even, \quad j=1,2,\dots,m \end{cases}$$

Case (ii) When n is even

$$f(v_i) = \begin{cases} (m+1)(n-1+i)+1 & when\ i\ is\ odd \\ (m+1)(3n-i)+1 & when\ i\ is\ even \end{cases}$$
$$f(v_{ij}) = \begin{cases} (m+1)(3n+1-i)-(2j-1) & when\ is\ odd, \quad j=1,2,\dots,m \\ (m+1)(n-2+i)+(2j+1) & when\ i\ is\ even, \quad j=1,2,\dots,m \end{cases}$$

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These vertex functions for odd and even n of $H_n \odot mK_1$in turn induce corresponding edge functions for odd and even n defined under f^*.
Define$f^*: E \to C \setminus f(V)$ by,

$$f^*(u_i u_{i+1}) = |i + m(i-1) - ((m+1)(4n - (i+1) + 1))|$$
$$= 2(m+1)(2n-i), i \text{ is odd}, \forall\, n$$
$$f^*(u_i u_{i+1}) = |(4n-i)(m+1) + 1 - ((i+1) + m(i+1-1))|$$
$$= 2(m+1)(2n-i), i \text{ is even}, \forall\, n$$
$$f^*(u_i u_{ij}) = |i + m(i-1) - ((m+1)(4n+1-i) - (2j-1))|$$
$$= 2((m+1)(2n+1-i) - j), i \text{ is odd}, j = 1,2,\dots,m, \forall\, n$$
$$f^*(u_i u_{ij}) = |(m+1)(4n-i) + 1 - ((m+1)(i-2) + (2j+1))|$$
$$= 2((m+1)(2n+1-i) - j), i \text{ is even}, j = 1,2,\dots,m, \forall\, n$$

Case (i) When n is odd

$$f^*(v_i v_{i+1}) = \left|(m+1)(3n-i) + 1 - \left((m+1)\big(n - 1 + (i+1)\big) + 1\right)\right|$$
$$= 2(m+1)(n-i), \qquad i \text{ is odd}$$
$$f^*(v_i v_{i+1}) = \left|(m+1)(n-1+i) + 1 - \left((m+1)\big(3n - (i+1)\big) + 1\right)\right|$$
$$= 2(m+1)(n-i), \qquad i \text{ is even}$$
$$f^*(v_i v_{ij}) = |(m+1)(3n-i) + 1 - ((m+1)(n-2+i) + (2j+1))|$$
$$= 2((m+1)(n+1-i) - j), \qquad i \text{ is odd}, \qquad j = 1,2,\dots,m$$
$$f^*(v_i v_{ij}) = |(m+1)(n-1+i) + 1 - ((m+1)(3n+1-i) - (2j-1))|$$
$$= 2((m+1)(n+1-i) - j), \qquad i \text{ is even}, \qquad j = 1,2,\dots,m$$
$$f^*\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\right) = \left|\left(\frac{n+1}{2}\right) + m\left(\frac{n+1}{2} - 1\right) - \left((m+1)\left(3n - \left(\frac{n+1}{2}\right)\right) + 1\right)\right|$$
$$= 2n(m+1), \qquad when\ \frac{n+1}{2} \text{ is odd}$$
$$f^*\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\right) = \left|(m+1)\left(4n - \left(\frac{n+1}{2}\right)\right) + 1 - \left((m+1)\left(n - 1 + \left(\frac{n+1}{2}\right)\right) + 1\right)\right|$$
$$= 2n(m+1), \qquad when\ \frac{n+1}{2} \text{ is even}$$

Case (ii) When n is even

$$f^*(v_i v_{i+1}) = \left|(m+1)(n-1+i) + 1 - \left((m+1)\big(3n - (i+1)\big) + 1\right)\right|$$
$$= 2(m+1)(n-i), \qquad i \text{ is odd}$$
$$f^*(v_i v_{i+1}) = \left|(m+1)(3n-i) + 1 - \left((m+1)\big(n - 1 + (i+1)\big) + 1\right)\right|$$
$$= 2(m+1)(n-i), \qquad i \text{ is even}$$
$$f^*(v_i v_{ij}) = |(m+1)(n-1+i) + 1 - ((m+1)(3n+1-i) - (2j-1))|$$
$$= 2((m+1)(n+1-i) - j), \qquad i \text{ is odd}, \qquad j = 1,2,\dots,m$$
$$f^*(v_i v_{ij}) = |(m+1)(3n-i) + 1 - ((m+1)(n-2+i) + (2j+1))|$$
$$= 2((m+1)(n+1-i) - j), \qquad i \text{ is even}, \qquad j = 1,2,\dots,m$$
$$f^*\left(u_{\frac{n}{2}+1} v_{\frac{n}{2}}\right) = \left|(m+1)\left(4n - \left(\frac{n+1}{2}\right)\right) + 1 - \left((m+1)\left(n - 1 + \frac{n}{2}\right) + 1\right)\right|$$
$$= 2n(m+1), \qquad when\ \frac{n}{2} \text{ is odd}$$
$$f^*\left(u_{\frac{n}{2}+1} v_{\frac{n}{2}}\right) = \left|\left(\frac{n}{2} + 1\right) + m\left(\frac{n}{2}\right) - \left((m+1)\left(3n - \frac{n}{2}\right) + 1\right)\right|$$
$$= 2n(m+1), \qquad when\ \frac{n}{2} \text{ is even}$$

On labeling the vertices and edges of $H_n \odot mK_1$ we see that all the members of C have been uniquely assigned without repetition under f and f^* respectively. Hence the definition of cardinal labeling is satisfied for both odd and even cases of n. So, we may conclude that $H_n \odot mK_1$, $m \geq 2$ admits cardinal labeling.

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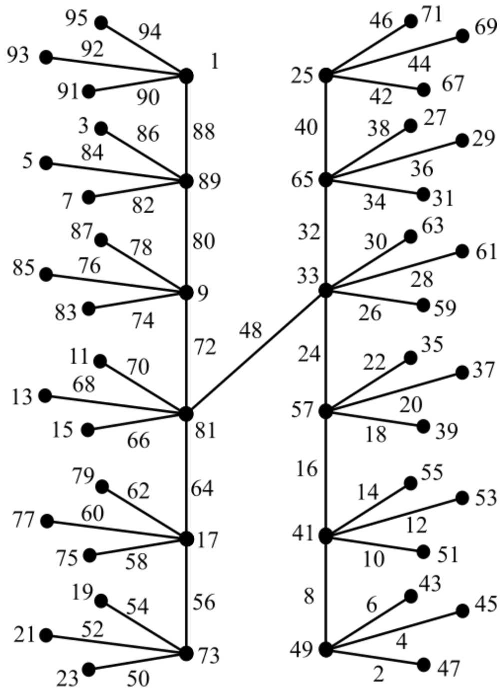

Figure 4: $\boldsymbol{H_6 \odot 3K_1}$

Theorem 2.5

Cardinal labeling is true for H-cracker graph $H_{n,k}$.

Proof

Let $u_1, u_2, \ldots, u_n$ *and* $v_1, v_2, \ldots, v_n$ denote the vertices of the two paths of H_n. Let u_i' *and* v_i' $, 1 \le i \le n$, denote the central vertices of the $2n$ stars $K_{1,k}$ attached to $u_i s$ and $v_i s$ respectively and $u_{ij}s$ and $v_{ij}s$ $(1 \le i \le n\,; 1 \le j \le k)$ denote their corresponding pendant vertices. H-cracker graph has $|V| = 2n(k+2), |E| = 2n(k+2) - 1$ and so $|C| = 4n(k+2) - 1$. We try to label the vertices of the graph cardinally by defining a function $f: V \to C$ by,

$$f(u_i) = \begin{cases} (k+2)\big(4n-(i+1)\big)+3, & when\ i\ is\ odd, \quad \forall\, n \\ (k+2)(i-2)+3, & when\ i\ is\ even, \quad \forall\, n \end{cases}$$

$$f(u_i') = \begin{cases} (k+2)(i-1)+1, & when\ i\ is\ odd, \quad \forall\, n \\ (k+2)(4n-i)+1, & when\ i\ is\ even, \quad \forall\, n \end{cases}$$

$$f(u_{ij}) = \begin{cases} (k+2)(4n+1-i)-(2j-1), & when\ i\ is\ odd, \quad \forall\, n \\ (k+2)(i-2)+(2j+3), & when\ i\ is\ even, \quad \forall\, n \end{cases}$$

Case (i) When n is odd

$$f(v_i) = \begin{cases} (k+2)(n-2+i)+3, & when\ i\ is\ odd \\ (k+2)\big(3n-(i+1)\big)+3, & when\ n\ is\ even \end{cases}$$

$$f(v_i') = \begin{cases} (k+2)(3n-i)+1, & when\ i\ is\ odd \\ (k+2)(n-1+i)+1, & when\ i\ is\ even \end{cases}$$

$$f(v_{ij}) = \begin{cases} (k+2)(n-2+i)+(2j+3), & when\ i\ is\ odd \\ (k+2)(3n+1-i)-(2j-1), & when\ i\ is\ even \end{cases}$$

Case (ii) When n is even

$$f(v_i) = \begin{cases} (k+2)\big(3n-(i+1)\big)+3, & when\ i\ is\ odd \\ (k+2)(n-2+i)+3, & when\ i\ is\ even \end{cases}$$

$$f(v_i') = \begin{cases} (k+2)(n-1+i)+1, & when\ i\ is\ odd \\ (k+2)(3n-i)+1, & when\ i\ is\ even \end{cases}$$

$$f(v_{ij}) = \begin{cases} (k+2)(3n+1-i)-(2j-1), & when\ i\ is\ odd \\ (k+2)(n-2+i)+(2j+3), & when\ i\ is\ even \end{cases}$$

Under f, all the vertices receive distinct members of $C. f$ induces $f^*: E \to C \backslash f(V)$ given by,

$$f^*(u_i u_{i+1}) = \begin{cases} \Big|(k+2)(4n-i-1)+3-\Big((k+2)\big((i+1)-2\big)+3\Big)\Big| = 2(k+2)(2n-i), \\ \qquad when\ i\ is\ odd, \quad \forall\, n \\ \Big|(k+2)(i-2)+3-\Big((k+2)\big(4n-(i+1+1)\big)+3\Big)\Big| = 2(k+2)(2n-i), \\ \qquad when\ is\ even, \quad \forall\, n \end{cases}$$

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$$f^*(u_iu_i') = \begin{cases} |(k+2)(4n-(i+1))+3-((k+2)(i-1)+1)| = 2((k+2)(2n-i)+1), \\ \text{when } i \text{ is odd}, \quad \forall n \\ |(k+2)(i-2)+3-((k+2)(4n-i)+1)| = 2((k+2)(2n+1-i)-1), \\ \text{when } i \text{ is even}, \quad \forall n \end{cases}$$

$$f^*(u_i'u_{ij}) = \begin{cases} |(k+2)(i-1)+1-((k+2)(4n+1-i)-(2j-1))| = 2((k+2)(2n+1-i)-j), \\ \text{when } i \text{ is odd}, \forall n \\ |(k+2)(4n-i)+1-((k+2)(i-2)+(2j+3))| = 2((k+2)(2n+1-i)-(j+1)), \\ \text{when } i \text{ is even}, \forall n \end{cases}$$

Case (i) When n is odd

$$f^*(v_iv_{i+1}) = \begin{cases} |(k+2)(n-2+i)+3-((k+2)(3n-(i+1+1))+3)| = 2(k+2)(n-i), \\ \text{when } i \text{ is odd} \\ |(k+2)(3n-(i+1))+3-((k+2)(n-2+i+1)+3)| = 2(k+2)(n-i), \\ \text{when } i \text{ is even} \end{cases}$$

$$f^*(v_iv_i') = \begin{cases} |(k+2)(n-2+i)+3-((k+2)(3n-i)+1)| = 2((k+2)(n+1-i)-1), \\ \text{when } i \text{ is odd} \\ |(k+2)(3n-(i+1)+3-((k+2)(n-1+i)+1)| = 2((k+2)(n-i)+1), \\ \text{when } i \text{ is even} \end{cases}$$

$$f^*(v_i'v_{ij}) = \begin{cases} |(k+2)(3n-i)+1-((k+2)(n-2+i)+(2j+3))| = 2((k+2)(n+1-i)-(j+1)), \\ \text{when } i \text{ is odd} \\ |(k+2)(n-1+i)+1-((k+2)(3n+1-i)-(2j-1))| = 2((k+2)(n+1-i)-j), \\ \text{when } i \text{ is even} \end{cases}$$

$$f^*\left(u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}\right) = \begin{cases} \left|(k+2)\left(4n-\left(\frac{n+1}{2}+1\right)\right)+3-\left((k+2)\left(n-2+\left(\frac{n+1}{2}\right)\right)+3\right)\right| = 2n(k+2), \\ \text{when } \frac{n+1}{2} \text{ is odd} \\ \left|(k+2)\left(\frac{n+1}{2}-2\right)+3-\left((k+2)\left(3n-\left(\frac{n+1}{2}+1\right)\right)+3\right)\right| = 2n(k+2), \\ \text{when } \frac{n+1}{2} \text{ is even} \end{cases}$$

Case (ii) When n is even

$$f^*(v_iv_{i+1}) = \begin{cases} |(k+2)(3n-(i+1))+3-((k+2)(n-2+i+1)+3)| = 2(k+2)(n-i), \\ \text{when } i \text{ is odd} \\ |(k+2)(n-2+i)+3-((k+2)(3n-(i+1+1))+3)| = 2(k+2)(n-i), \\ \text{when } i \text{ is even} \end{cases}$$

$$f^*(v_iv_i') = \begin{cases} |(k+2)(3n-(i+1)+3-((k+2)(n-1+i)+1)| = 2((k+2)(n-i)+1), \\ \text{when } i \text{ is odd} \\ |(k+2)(n-2+i)+3-((k+2)(3n-i)+1)| = 2((k+2)(n+1-i)-1), \\ \text{when } i \text{ is even} \end{cases}$$

$$f^*(v_i'v_{ij}) = \begin{cases} |(k+2)(n-1+i)+1-((k+2)(3n+1-i)-(2j-1))| = 2((k+2)(n+1-i)-j), \\ \text{when } i \text{ is odd} \\ |(k+2)(3n-i)+1-((k+2)(n-2+i)+(2j+3))| = 2((k+2)(n+1-i)-(j+1)), \\ \text{when } i \text{ is even} \end{cases}$$

$$f^*\left(u_{\frac{n}{2}+1}v_{\frac{n}{2}}\right) = \begin{cases} \left|(k+2)\left(\frac{n}{2}+1-2\right)+3-\left((k+2)\left(3n-\left(\frac{n}{2}+1\right)\right)+3\right)\right| = 2n(k+2), \text{when } \frac{n}{2} \text{ is odd} \\ \left|(k+2)\left(4n-\left(\frac{n}{2}+1+1\right)\right)+3-\left((k+2)\left(n-2+\frac{n}{2}\right)+3\right)\right| = 2n(k+2), \\ \text{when } \frac{n}{2} \text{ is even} \end{cases}$$

Once all the edges are given the suitable labels, it could be observed that both the vertex and edge labels of $H_{n,k}$ are unique members of C. This leads us to a conclusion that H-cracker graph admits cardinal labeling.

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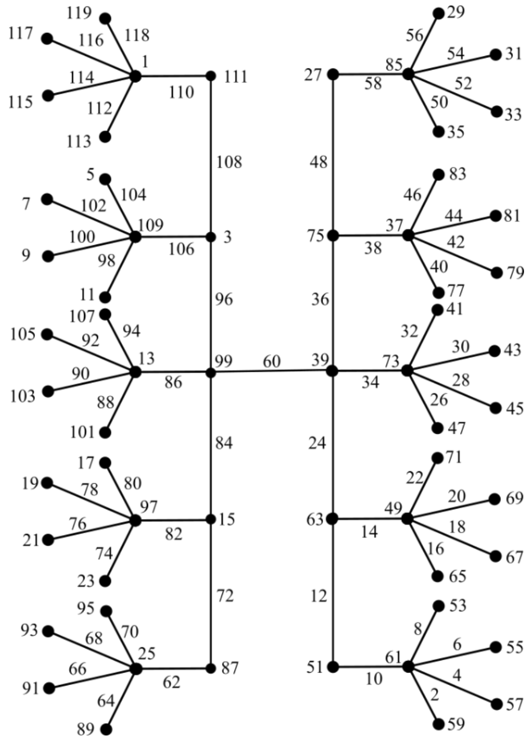

Figure 5: $\boldsymbol{H_{5,4}}$

3 CONCLUSION

In this work we have discussed cardinal labeling for H-graph, subdivided H-graph $S(H_n)$, corona of H-graph $H_n \odot K_1$, $H_n \odot mK_1$ and H-cracker graph $H_{n,k}$. Extending this work to various other graph operations related to H-graphs could be worked on by the reader.

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