REDVET - Revista electrónica de Veterinaria - ISSN 1695-7504
Vol 25, No. 1 (2024)
http://www.veterinaria.org
Article Received 05/08/2024 Revised 12/09/2024 Accepted 07/10/2024

Determinant Of Theta Type Bicyclic Graphs Using Incycle Vertices

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Abstract: The determinant of the adjacency matrix of the graph is generally called the determinant of the graph. Finding the determinant of graphs has been a topic of interest in algebraic graph theory. Many methods of reduction and formulae have been devised over the years. One of the formulae given by F. Harary, to find the determinant of the graph requires finding all the sesquivalent spanning subgraphs of a graph. The concept of incycle vertices depends on sesquivalent spanning subgraphs. We have used incycle vertices to find the determinant of bicyclic graphs with a common edge in this paper.

Keywords: Sesquivalent, bicyclic graph, determinant, incycle vertex.

1 INTRODUCTION

A graph is called a sesquivalent graph if all its components are cycles or edges. A spanning subgraph that is sesquivalent is called a sesquivalent spanning subgraph[1]. Perfect matching is also a sesquivalent spanning subgraph. The determinant of a graph G whose adjacency matrix is given by $A(G)$ is given by

$$det\big(A(G)\big) = \sum(-1)^{r(\Gamma)}(2)^{s(\Gamma)}$$

where $c(\Gamma)$ is the number of components of Γ, $r(\Gamma) = |V(\Gamma)| - c(\Gamma)$ and $s(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + c(\Gamma)$ and the summation is over all sesquivalent spanning subgraphs Γ of G[2]. This formula was given by Harary F, in 1962. The concept of incycle vertices depends on sesquivalent spanning subgraphs of a graph.
Gong et al have worked on finding the determinant of distance matrix of bicyclic graphs[3]. Jianxi et al have studied the nullity of bicyclic graphs[4]. Ma et al and Supot Sookyang et al have characterised some non-singular cyclic graphs[5][6]. In this paper we attempt to find the determinant of theta type bicyclic graphs using the concept of incycle vertices.

2 PRELIMINARIES

A vertex is called an *incycle vertex* if it lies on some cycle in all possible sesquivalent spanning subgraphs of a graph, except the perfect matching.
The graph got from identifying an edge of one cycle with an edge of another cycle, as $e = v_1v_2$, is a type of θ graph[4]. We denote the graph with trees attached to this type of θ graph by θ_n^1, where n is the number of vertices. In other words, θ_n^1 denotes the graph with two cycles having a common edge and trees attached to it, on n vertices.

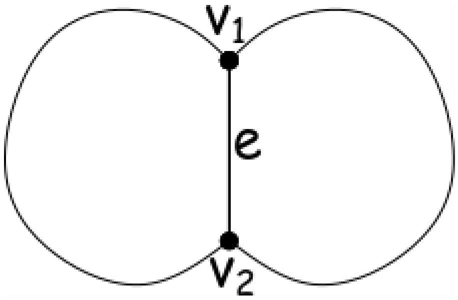

Figure 2.1 **–** A theta type bicyclic graph

Theorem 2.1: If $G \in \theta_n^1$ has an incycle vertex that is not in the common edge then the incycle vertex is not unique. In particular all vertices of that cycle are incyclic.
Proof: Suppose a vertex of a cycle other than v_1, v_2 is incycle vertex then it lies on a cycle in all sesquivalent spanning subgraphs of G, except the perfect matching. Hence that cycle is in every sesquivalent spanning subgraph other than the perfect matching. Therefore all its vertices including v_1, v_2 are incyclic.
Theorem 2.2: If $G \in \theta_n^1$ has no incycle vertex then the perfect matching is the only possible sesquivalent spanning subgraph and $|det\ (A(G))| \leq 1$.

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Proof: If G has a sesquivalent spanning subgraph with cycle in it then at least $v_1$and v_2 is an incycle vertex. If v_1 and v_2 are not incycle vertices then G has no incycle vertices. Hence the perfect matching is the only sesquivalent spanning subgraph. If G has a sesquivalent spanning subgraph then its determinant is either -1 or 1. If G has no sesquivalent spanning subgraphs then its determinant is 0.
Corollary 2.3: If $G \in \theta_n^1$ has no incycle vertices then determinant is 0, 1 or -1
Corollary 2.4: Let $G \in \theta_n^1$ and let G have no incycle vertices. Then G is singular if n is odd.
Theorem 2.5: Let $G \in \theta_n^1$ and let G have no incycle vertices. Then G is singular iff it has no sesquivalent spanning subgraph.
Proof: Let G be singular. From theorem 4.2 G has at most one sesquivalent spanning subgraph. If G has a perfect matching then G is not singular. The converse is obvious.

3 DETERMINANT OF BICYCLIC GRAPHS WITH COMMON EDGE
Theorem 3.1: Let $G \in \theta_n^1$ and let $G - \{v_1, v_2\}$ have a perfect matching. Then G has incycle vertices iff all vertices of at least one cycle are matched within itself.
Proof: Let G have incycle vertices. Suppose both cycles have at least one vertex matched with a vertex not in cycle. Then it is not possible to have any sesquivalent spanning subgraph with cycle. Therefore G has no incycle vertices. Conversely, let at least one cycle be matched within itself. Then all the vertices of that cycle together with v_1, v_2 forms a cyclic component in sesquivalent spanning subgraph of G. Hence G has incycle vertices.
Corollary 3.2: Let $G \in \theta_n^1$ and let $G - \{v_1, v_2\}$ have a perfect matching in which no cycle is matched within itself. Then G has only one perfect matching and $|det\ (A(G))| = 1$.
Theorem 3.3: Let $G \in \theta_n^1$ and let G have no incycle vertices. G is non - singular iff $G - \{v_1, v_2\}$ has a perfect matching or there exists a maximum matching of $G - \{v_1, v_2\}$ in which exactly one neighbour of v_1 and one neighbour of v_2 are unmatched.
Proof: Let G be non-singular. G has no incycle vertex. Hence by theorem 4.2, G has only one perfect matching. In the perfect matching either $v_1$and v_2 are matched with each other or they are matched with their neighbours, which leads to either of the cases. The converse is obvious.
Theorem 3.4: Let $G \in \theta_n^1$ have incycle vertices and let $G - \{v_1, v_2\}$ have no perfect matching. Then G is non-singular iff $G - \{v_1, v_2\}$ has a maximum matching in which exactly one vertex in a cycle is unmatched and all vertices of that cycle are matched within itself.
Proof: Let G be non-singular. Therefore G has at least one sesquivalent spanning subgraph with a cycle. If G has an even cycle in its sesquivalent spanning subgraph then $G - \{v_1, v_2\}$ has a perfect matching. Therefore G has an odd cycle in its sesquivalent spanning subgraph. This implies that $G - \{v_1, v_2\}$ has a maximum matching in which exactly one vertex in a cycle is unmatched and all vertices of that cycle are matched within itself.
Conversely, let $G - \{v_1, v_2\}$ have a maximum matching in which exactly one vertex in a cycle unmatched and all vertices of that cycle are matched within itself. Then G has a sesquivalent spanning subgraph with an odd cycle and hence it is non-singular.
Corollary 3.5: Let $G \in \theta_n^1$ and let $G - \{v_1, v_2\}$ have no perfect matching. Then G is non-singular iff G satisfies either one of following cases.
i) $G - \{v_1, v_2\}$ has a maximum matching in which exactly one neighbour of v_1 and one neighbour of v_2 are unmatched.
ii) $G - \{v_1, v_2\}$ has a maximum matching in which exactly one vertex in a cycle is unmatched and all vertices of that cycle are matched within itself. G has odd number of incycle vertices and $|\det\ A(G)| = 2$.
Corollary 3.6: Let $G \in \theta_n^1$. G is singular if more than 2 vertices are unmatched in the maximum matching of $G - \{v_1, v_2\}$.
Corollary 3.7: Let $G \in \theta_n^1$ and let $G - \{v_1, v_2\}$ have no perfect matching. The maximum matching of G has one unmatched vertex. G is singular if $G - \{v_1, v_2\}$ does not have a maximum matching in which the unmatched vertex lies on an odd cycle of G.
The proof follows from case (ii) of corollary 4.10.
Theorem 3.8: Let $G \in \theta_n^1$. If $G - \{v_1, v_2\}$ has a perfect matching then G cannot have odd number of incycle vertices.
Proof: Suppose G has an odd number of incycle vertices. Then there is an odd cycle in its sesquivalent spanning subgraph. Therefore $G - \{v_1, v_2\}$ does not have a perfect matching which is not possible. Hence G does not have odd number of incycle vertices.
Lemma 3.9: Let $G \in \theta_n^1$ be a graph with 2 incycle vertices. Then $|\det\ A(G)|$ is either 1, 3 or 7.
Proof: For G to have 2 incycle vertices it has to have 2 sesquivalent spanning subgraphs with cycles. There are 3 possibilities in which G has 2 incycle vertices. In all of these possibilities both the cycles in G have to be even.
Case i) If both cycles are of length $4k, k \in \mathbb{N}$, G has 3 perfect matchings, contributing $(-1)^{\frac{n}{2}}$ and 2 sesquivalent spanning subgraphs with cycles, contributing $(-1)^{\frac{n}{2}+1} \times 2$ each to determinant. Hence $\det A(G) = 3 \times (-1)^{\frac{n}{2}} + 2 \times (-1)^{\frac{n}{2}+1} \times 2 = (-1)^{\frac{n}{2}}[3 - 4] = (-1)^{\frac{n}{2}+1}$.

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Case ii) If both cycles are of length $4k+2, k \in \mathbb{N}$, G has 3 perfect matchings, contributing $(-1)^{\frac{n}{2}}$ and 2 sesquivalent spanning subgraphs with cycles, contributing $(-1)^{\frac{n}{2}} \times 2$ each to determinant. Hence $det\left(A(G)\right) = (-1)^{\frac{n}{2}} + 2 \times (-1)^{\frac{n}{2}} \times 2 = (-1)^{\frac{n}{2}}[3+4] = (-1)^{\frac{n}{2}} \times 7$.
Case iii) If one cycle is of length $4k$ and another is of length $4k+2, k \in \mathbb{N}$, G has 3 perfect matchings, contributing $(-1)^{\frac{n}{2}}$ and 2 sesquivalent spanning subgraphs with cycles, where one contributes $(-1)^{\frac{n}{2}} \times 2$ and another contributes $(-1)^{\frac{n}{2}+1} \times 2$ each to determinant. Hence $det\left(A(G)\right) = 3 \times (-1)^{\frac{n}{2}} + (-1)^{\frac{n}{2}+1} \times 2 + (-1)^{\frac{n}{2}} \times 2 = (-1)^{\frac{n}{2}} \times 3$.

Theorem 3.10: Let $G \in \theta_n^1$ and let $G - \{v_1, v_2\}$ have a perfect matching. G is singular iff the number of incycle vertices is $4k, k \in \mathbb{N}$.

Proof: Let G be singular. If G has no incycle vertices, then it is non-singular. If G has an odd number of incycle vertices then also G is non-singular. If G has $4k+2, k \in \mathbb{N}$ incycle vertices then the determinant is $(-1)^{\frac{n}{2}+1}4$. By lemma 4.14, determinant is non zero if the number of incycle vertices is 2. Hence the number of incycle vertices is $4k, k \in \mathbb{N}$.
Let G have $4k, k \in \mathbb{N}$ incycle vertices. Then G has one sesquivalent spanning subgraph with cycle and 2 perfect matchings. The determinant is zero and hence G is singular.

Theorem 3.11: Let $G \in \theta_n^1$. G has no incycle vertex if more than 1 vertex is unmatched in the maximum matching of $G - \{v_1, v_2\}$.

Proof: If there are more than 2 unmatched vertices in the maximum matching then G has no sesquivalent spanning subgraph and hence no incycle vertex, by corollary 4.11. For G to have incycle vertices, it has to have at least one sesquivalent spanning subgraph with cycle. All the vertices of at least one cycle has be matched within itself, except the unmatched vertices. Now if G has 2 unmatched vertices, then the two vertices do not lie on the same cycle and hence G has no incycle vertex. Hence the theorem.

Theorem 3.12: Let $G \in \theta_n^1$ with both cycles even and let $G - \{v_1, v_2\}$ have no perfect matching. G is singular if the number of unmatched vertices in a maximum matching of $G - \{v_1, v_2\}$ is not equal to 2.

Proof: Suppose G has more than 2 unmatched vertices in a maximum matching of $G - \{v_1, v_2\}$ then G has no sesquivalent spanning subgraphs and hence no incycle vertices. Consider the case where $G - \{v_1, v_2\}$ has exactly one unmatched vertex. If the unmatched vertex lies on a cycle then at least one vertex of that cycle is matched outside the cycle. G has no sesquivalent spanning subgraphs and hence is singular.

Theorem 3.13: Let $G \in {\theta_n}^1$ with one odd cycle and one even cycle and, let $G - \{v_1, v_2\}$ have no perfect matching. G has no incycle vertices if the maximum matching of $G - \{v_1, v_2\}$ has more than one unmatched vertices. The vertices of the even cycle other than v_1 and v_2 cannot be incycle.

Proof: If $G - \{v_1, v_2\}$ has more than 2 unmatched vertices then G is has no sesquivalent spanning subgraph. Suppose $G - \{v_1, v_2\}$ has 2 unmatched vertices. If both vertices lie on the even cycle, G has at most one perfect matching and hence no incycle vertices. If both vertices lie on the odd cycle, at least one vertex has been matched outside the cycle and hence G has at most one perfect matching and no incycle vertices. If one vertex lies on each cycle then also G has no sesquivalent spanning subgraphs with cycles. If at least one vertex lies outside both cycles, then G has at most one perfect matching. Hence G has no incycle vertex.
Let $G - \{v_1, v_2\}$ have one vertex unmatched. If the unmatched vertex lies on the odd cycle and if all the vertices of the cycle are matched within itself then G has sesquivalent spanning subgraph with odd cycle. Hence all vertices of the odd cycles are incycle and no vertex of the even cycle other than v_1 and v_2 are incyclic.

Corollary 3.14: Let $G \in \theta_n^1$ and let the maximum matching of $G - \{v_1, v_2\}$ have one unmatched vertex. G has no incycle vertices if G has no odd cycles.

Remark 3.15: Let $G \in \theta_n^1$.

i) If both cycles are odd, then vertices of both cycles can be incycle vertices. The number of incycle vertices cannot be 2.
ii) If both cycles are even, then vertices of at most one cycle can be incycle vertices. The number of incycle vertices is even.
iii) If one cycle is odd and one cycle is even, then vertices of at most one cycle can be incycle vertices. The number of incycle vertices cannot be 2.

In general, if G has an even cycle then the vertices of at most one cycle can be incycle vertex. If G has an odd cycle the number of incycle vertices cannot be 2. If G has 2 odd cycles there is at most one sesquivalent spanning subgraph with cycles.

4 CONCLUSION

In this paper we have discussed about the determinant of θ type bicyclic graphs. We have deduced their determinants using the concept of incycle vertices. Our future topics of interest include finding the determinant of multicyclic graphs with a common edge and trees attached to it. By using the reduction procedures given by Rara we can also deduce the determinant of graphs having cycles with $4k+1$, $k \in \mathbb{N}$ common edges [7].

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